Exploratory and Confirmatory Factor models: Continued

Factors as models of data

Weighting items as summaries of data

I. Conceptually, combine items/variables into weighted linear composites to allow simpler description of data

II. Scale scores are just unit weighted items

III. Component scores are "optimally weighted"

IV. Factor scores are estimates of scores based upon a latent model

Components as data summaries

I. Data level

A. Components = XW or $CW^{-1} = X$

1. X is the matrix of data

2. W is a matrix of component weights

B. ${}_nC_m = {}_nX_m \, {}_mW'_m$

C. This is a rotation of the data to allow for composite summaries

D. Weights can be -1, 0, 1 (item composites)

E. Weights can be found to maximize variance of C

Principal Components as summaries of data

I. Principal Components are those weighted components such that successive components have maximum variance

II. Structure Level

A. Let X be zero centered (i.e., subtract means)

B. $Cov = X'X/(n-1) = CC'$

C. ${}_mCov_m = {}_mX'_n\, {}_nX_m /(n-1) = {}_mC_m\, {}_mC'_m$

D. If $k < m$ then we can approximate ${}_mCov_m \approx {}_mC_k\, {}_kC'_m$

Factors as models of data

I. Data level

A. $X \approx S F + E$

1. S is the matrix of factor scores
2. F is a matrix of factor loadings
3. E is a matrix of residuals

B. ${}_nX_m \approx {}_nS_k\, {}_kF'_m + {}_nE_m$

C. This is a regression model with two unknowns, the factor scores and the factor weights

Factors as models of data

I. Structure Level

A. Let X be zero centered (i.e., subtract means)

B. $C = X'X/(n-1) = FF'$

C. ${}_mC_m = {}_mX'_n\,{}_nX_m\,/(n-1) \approx {}_mF_k\,{}_kF'_m + U^2$

Determining communalities

I. Communality of an item is amount of variance that all of the factors account for in that item

II. If we know the communalities, then extracting the factors is straightforward (eigen value decomposition), but what is the communality?

Alternative Communality estimates

I. Highest correlation

II. Squared multiple R (SMC) = 1 - diag(R^{-1})

III. Iterative

   A. initial estimate

   B. extract F, then find FF' and use those values

   C. extraction using eigen value decomposition

IV. Anything (for larger problems)

The number of factors

I. Kaiser-Guttman rule: eigen value > 1

   A. logic is that a component should account for more variance than a single variable

   B. tends to suggest nv/3 and not be sensitive to the data

   C. tends to be the default for many programs (but not R)

Number of factors: continued

I. scree test

II. Parallel analysis

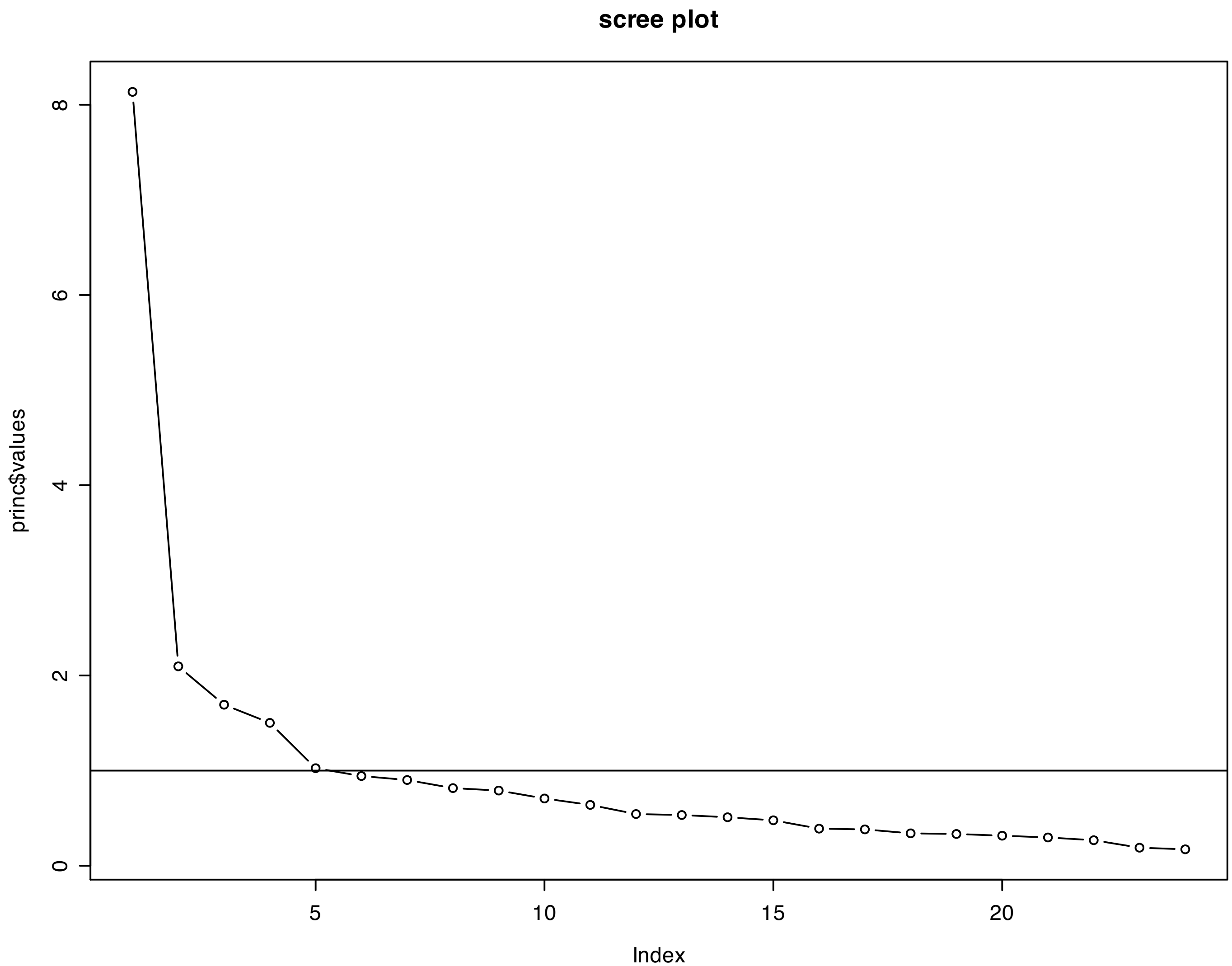
III. Velicer's MAP (minimum average partial)

IV. Revelle/Rocklin Very Simple Structure

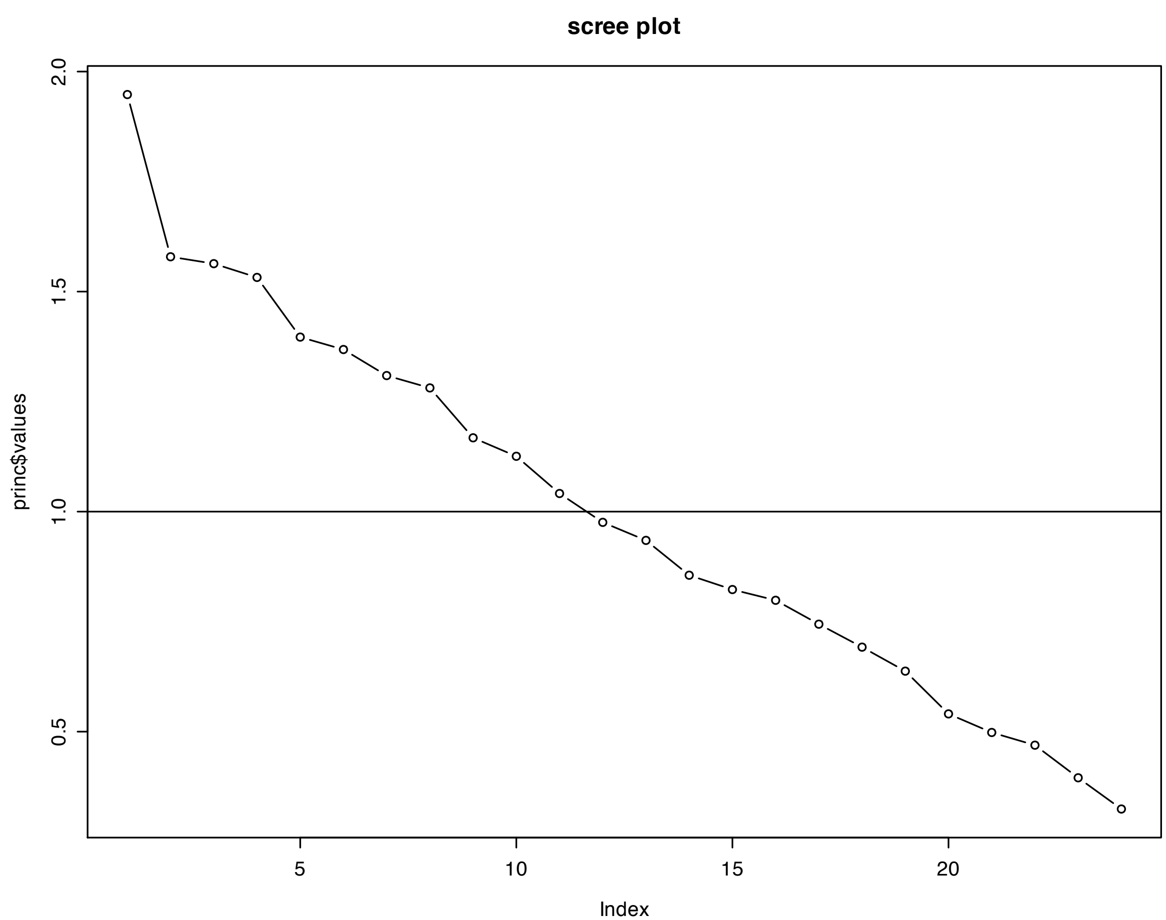
   A. designed for low complexity problems

   B. (e.g., items with simple structure, perhaps circumplex structure)

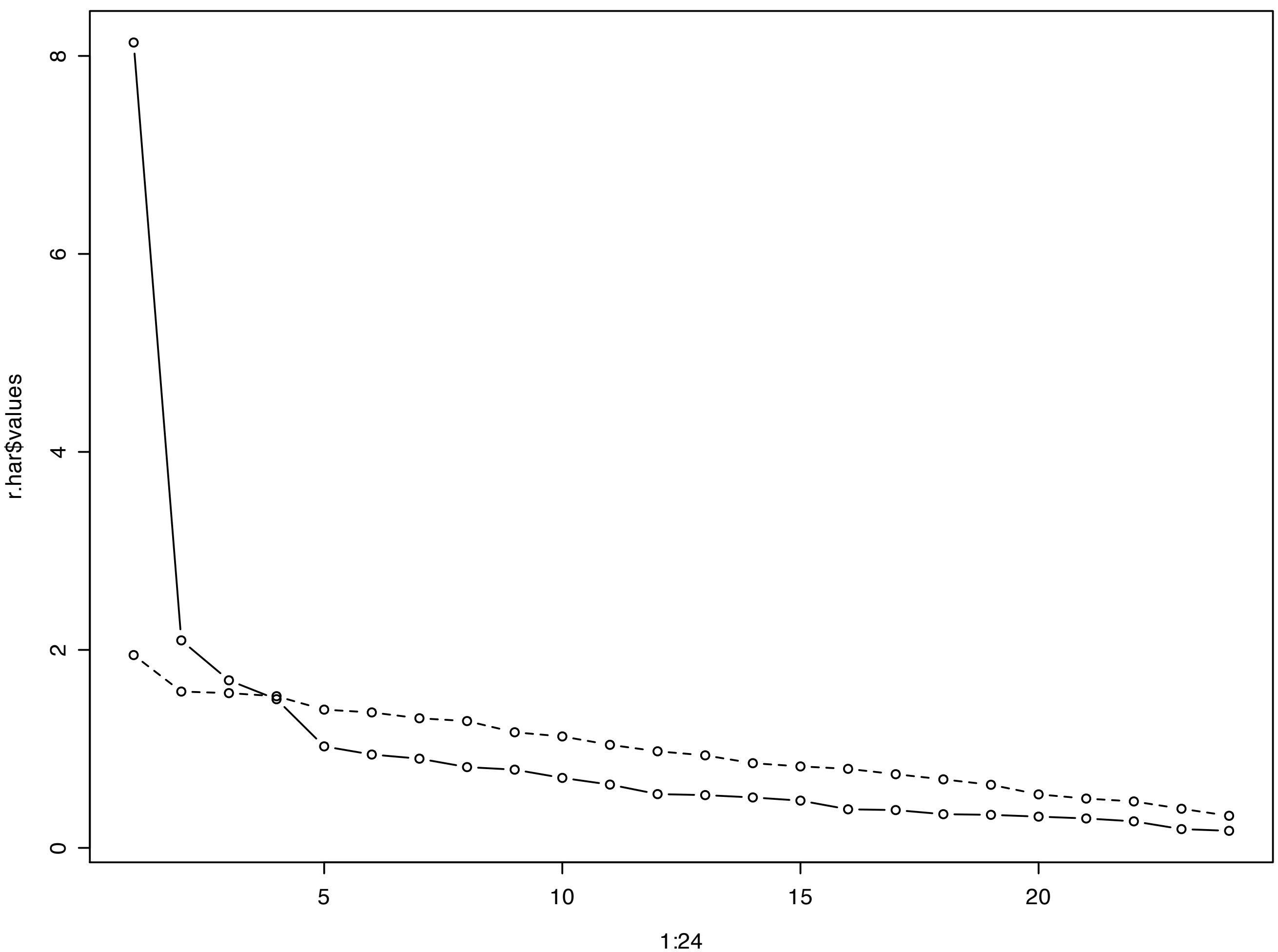
Harmon's 24 variables

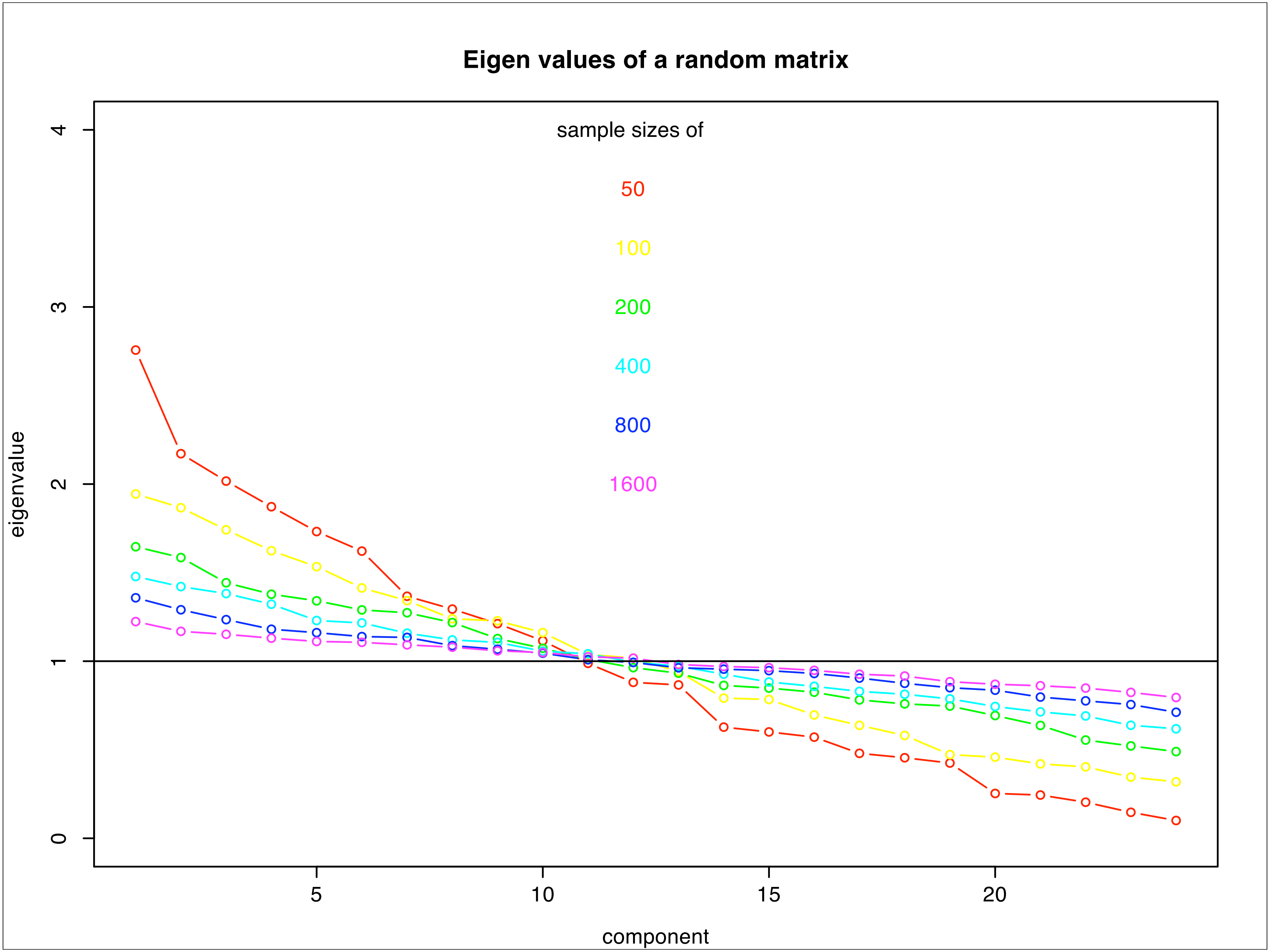


Random 24



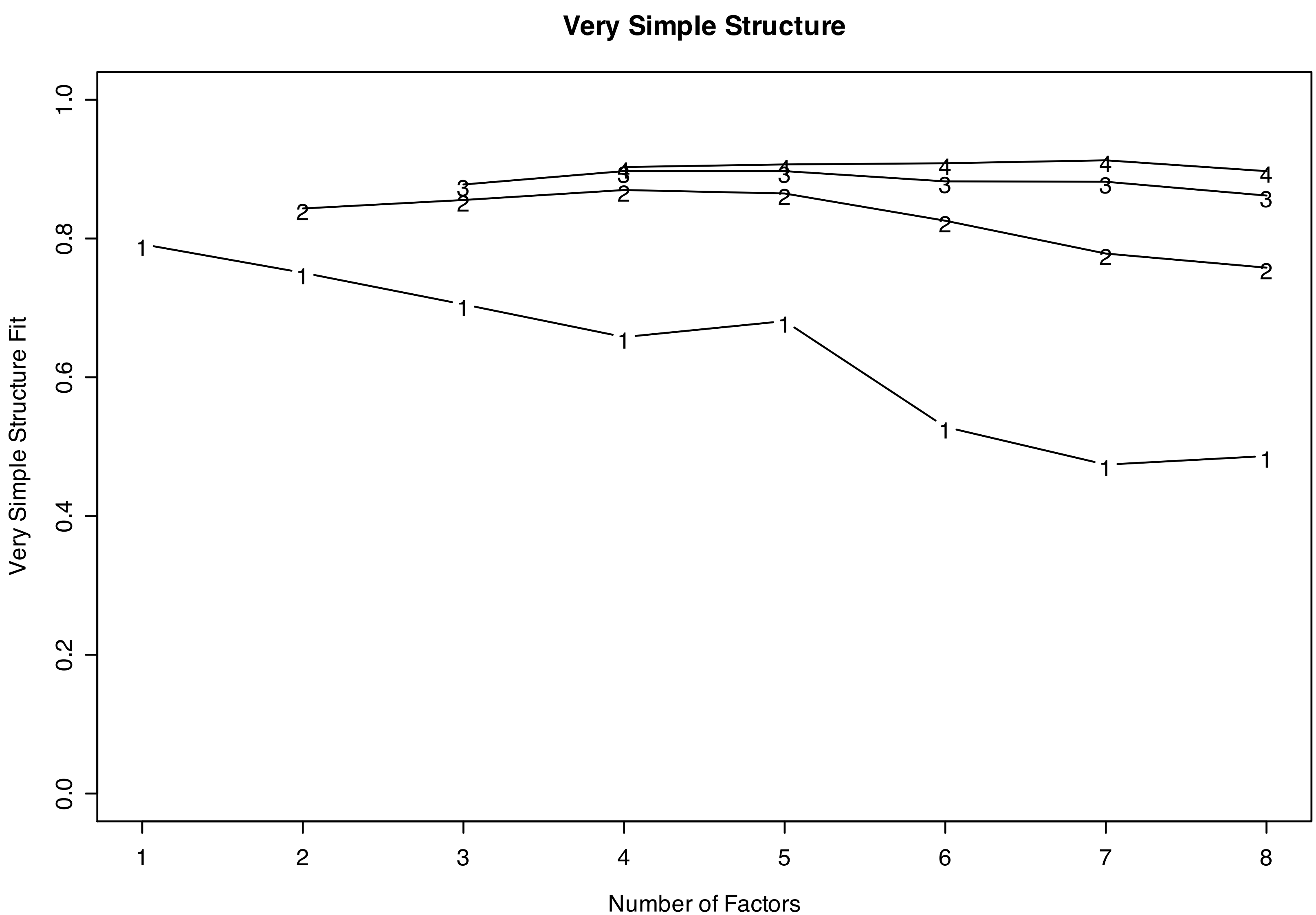
Real vs. Random





Eigen values of a random matrix
sample sizes of
50
100
200
400
800
1600
eigenvalue
component
0
1
2
3
4
5
10
15
20

Very Simple Structure



Plots of scree

```
test.data <- Harman74.cor$cov
VSS.scree(test.data)

r.data <- cor(matrix(rnorm(24*145),ncol=24))
VSS.scree(r.data)

r.pc <- principal(r.data,24)
r.har <- principal(test.data,24)
plot(1:24,r.har$values,type="b")
points(1:24,r.pc$values,type="b",lty="dashed")

VSS.plot(VSS(test.data))
```

Rotation:Transformation

I. Factors as extracted are in order of variance accounted for.

   A. optimal ordering for accounting for variance of covariance/correlation matrix

   B. non-optimal for interpretability

II. Rotation to “Simple Structure”

What is simple structure?

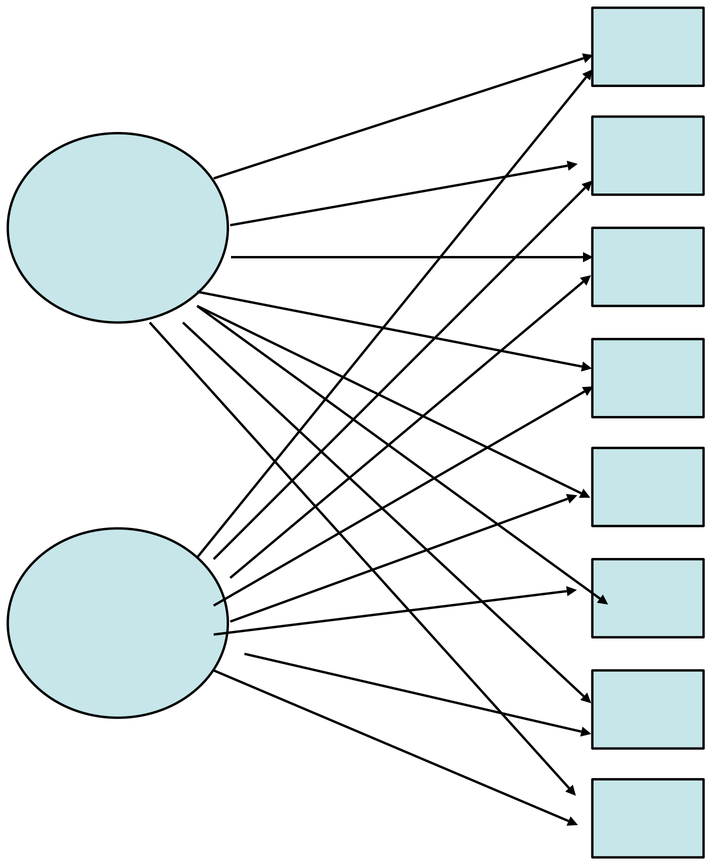
I. Simple in the eye of the beholder

II. Minimize number of paths

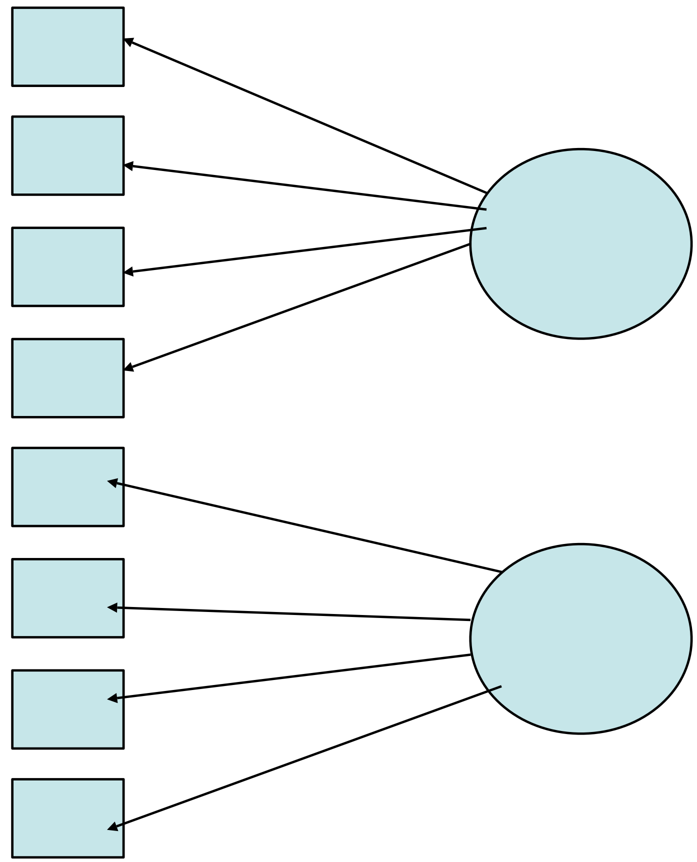
III. Maximize number of zero (or very small) loadings

IV. Orthogonal? Oblique?

Rotation to simple structure
Original

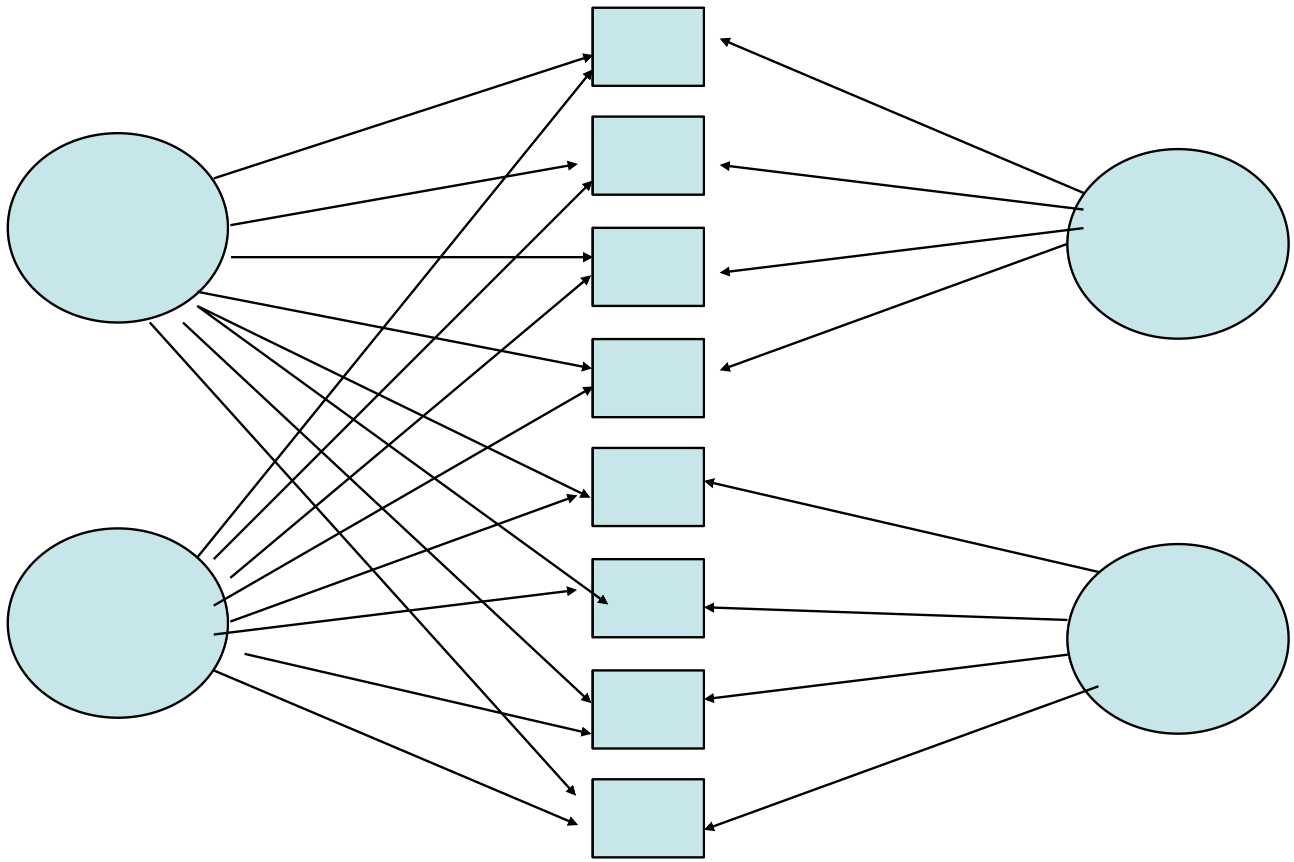


Rotation to simple structure
Simple Structure



Rotation to simple structure

Original Rotated



Orthogonal rotations

I. Find a T such that

A. $F_t = T F$

B. TT' = I (orthogonal rotations)

C. some function of F_t is maximized

II. Quartimax: $\sum\sum p^4$ = where p = loading

A. Maximizes variance of squared loadings

III. VARIMAX = $\sum\sum p^4 - (1/k)\sum_f(\sum_v p^2)^2$

A. maximizes column (factor) variance of loadings

Orthogonal rotations

I. VARIMAX tends to equalize factors and will obscure a general factor

II. Quartimax tends to produce general factors

III. Neither tend to produce bi-factor solutions

IV. Hand rotations can be done easily

Why hand rotate?

I. Fun?

II. Maximize a particular criterion not available in standard packages

III. Create new criteria for structural test

   A. e.g. Acton and Revelle tests for circumplex structure

   B. http://personality-project.org/r/circumplex.html

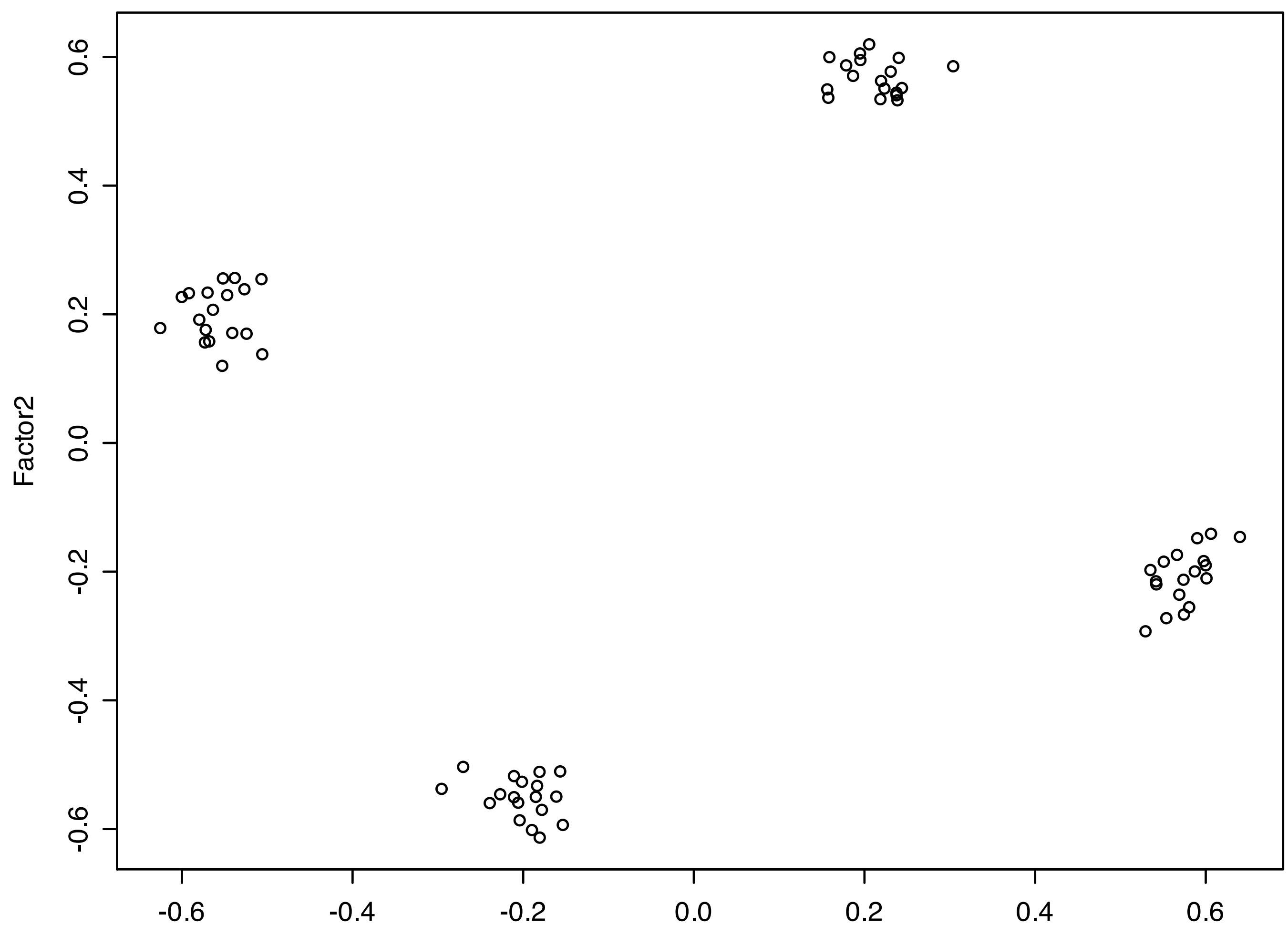
Hand Rotation

cos(ø)	0...0	-sin(ø)
0 ... 0	1...1	0 ... 0
-sin(ø)	0...0	cos(ø

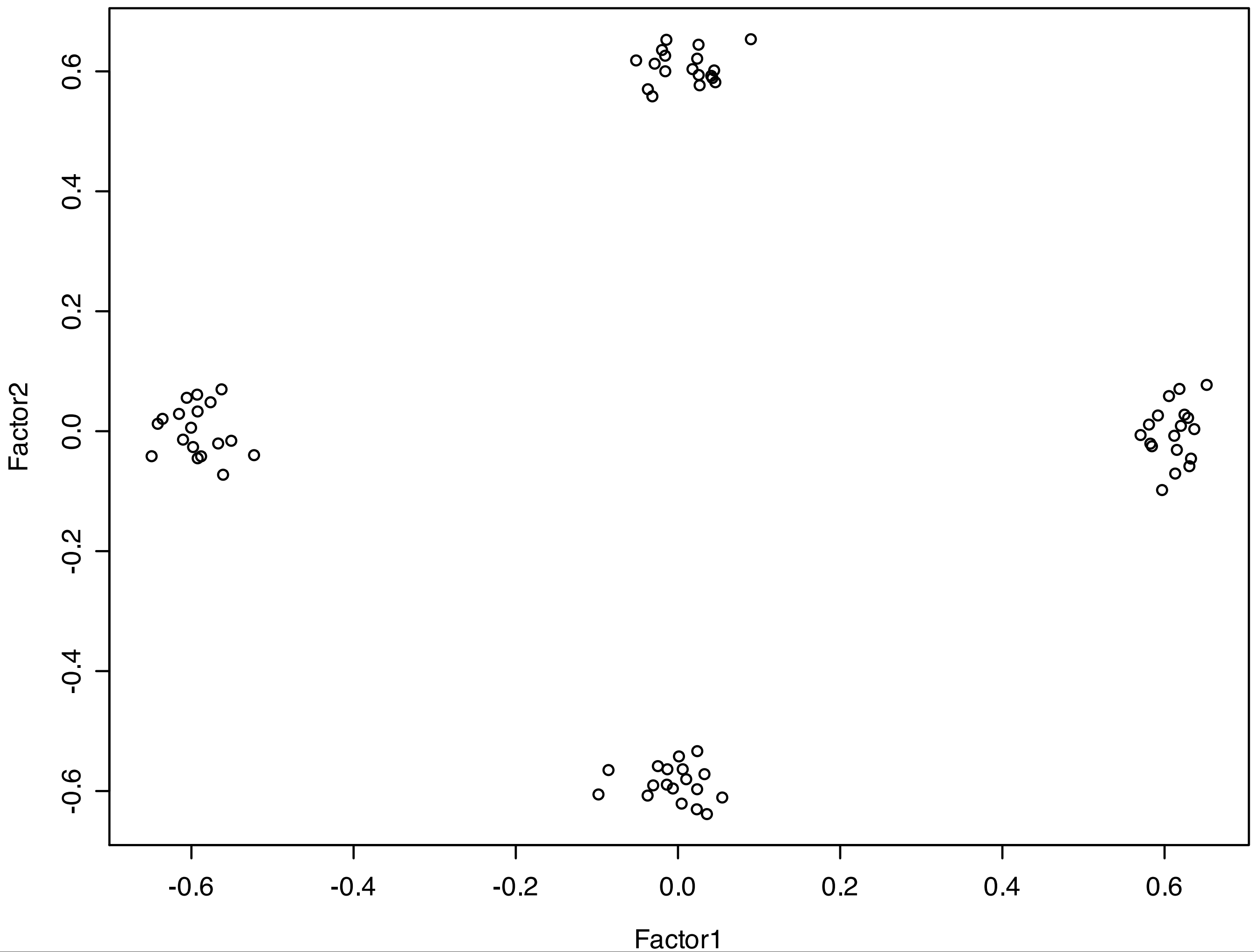
Hand rotation

```
function(f,angle,col1,col2)  {
   #hand rotate two factors from a loading matrix
   #see the GPArotation package for much more elegan
procedures
     nvar<- dim(f)[2]
     rot<- matrix(rep(0,nvar*nvar),ncol=nvar)
     rot[cbind(1:nvar, 1:nvar)] <- 1
     theta<- 2*pi*angle/360
     rot[col1,col1]<- cos(theta)
     rot[col2,col2]<- cos(theta)
     rot[col1,col2]<- -sin(theta)
     rot[col2,col1]<- sin(theta)
     result <- f %*% rot
     return(result) }
```

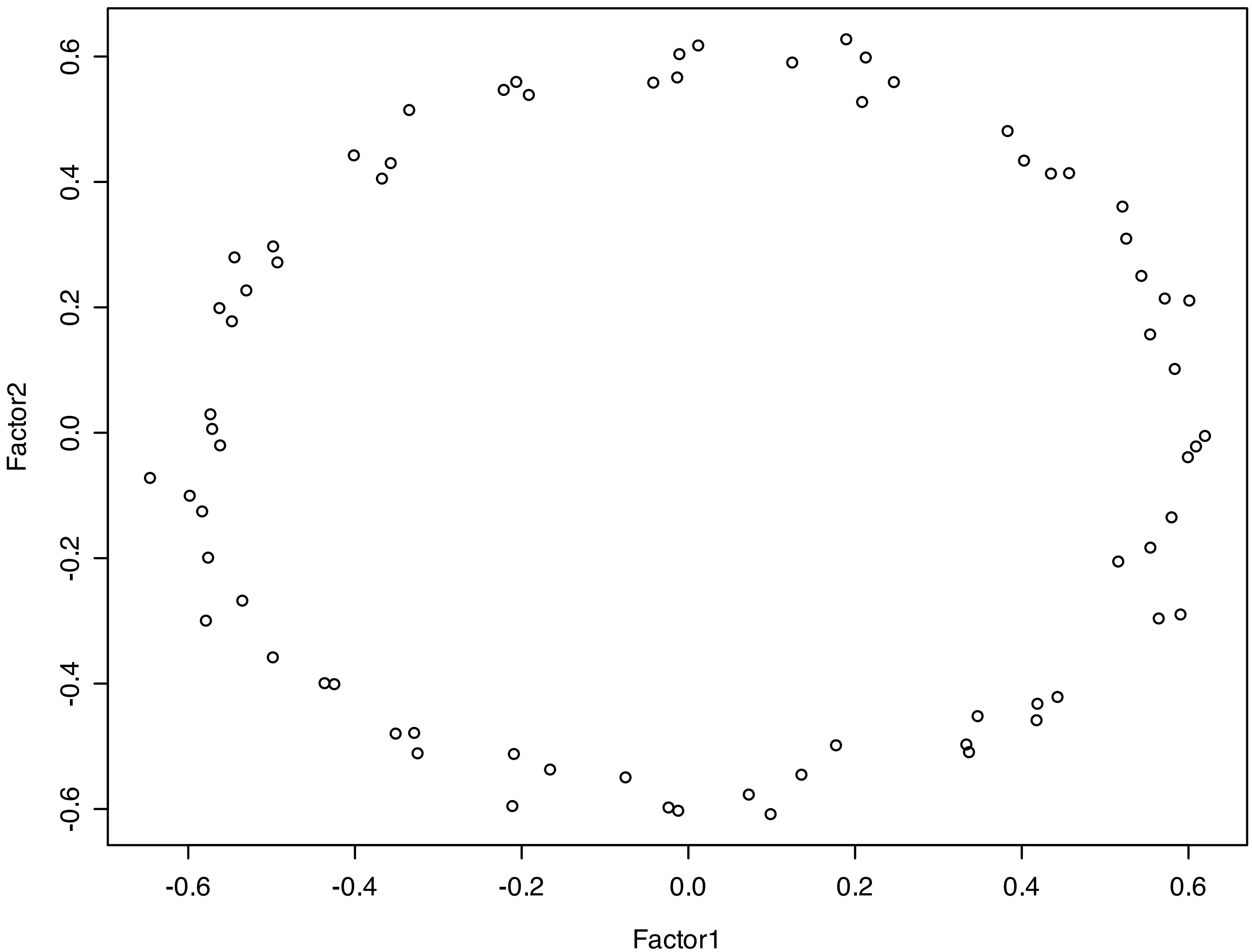
Simple data, not rotated



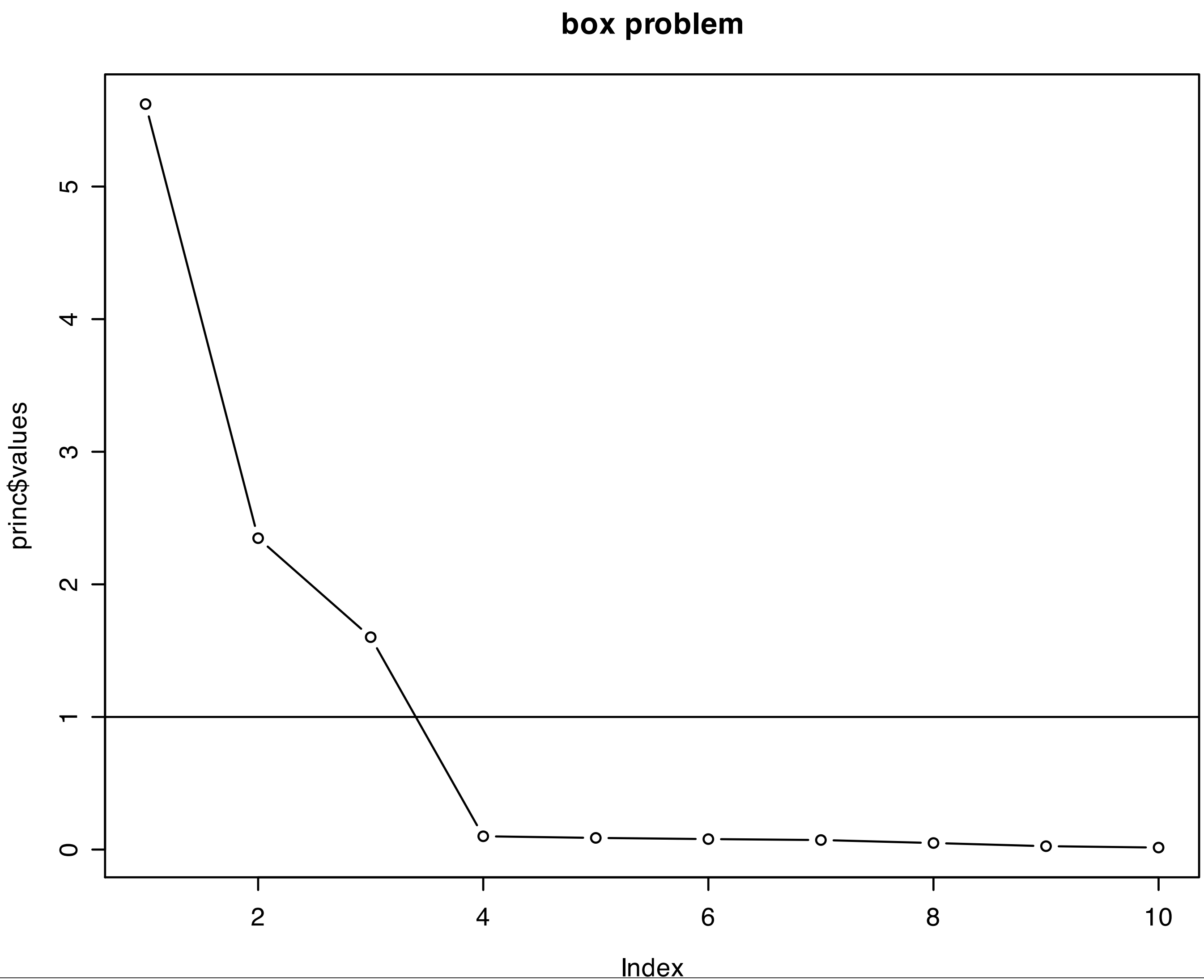
Simple Structure



Circumplex structure



Dimensions of boxes



Factors of boxes

	Factor1	Factor2	Factor3
V1	0.08	0.06	0.96
V2	0.10	0.94	0.16
V3	0.95	0.08	-0.02
V4	0.10	0.77	0.60
V5	0.89	0.05	0.36
V6	0.84	0.50	0.05
V7	0.17	0.22	0.92
V8	0.20	0.95	0.02
V9	0.95	0.02	0.02
V10	0.79	0.43	0.39

Orthogonal vs. Oblique

	Factor1	Factor2	Factor3		Factor1	Factor2	Factor3
V1	0.08	0.06	0.96	V1	0.00	−0.08	0.99
V2	0.10	0.94	0.16	V2	−0.03	0.96	0.03
V3	0.95	0.08	−0.02	V3	0.99	−0.05	−0.12
V4	0.10	0.77	0.60	V4	−0.05	0.72	0.51
V5	0.89	0.05	0.36	V5	0.89	−0.12	0.29
V6	0.84	0.50	0.05	V6	0.81	0.40	−0.09
V7	0.17	0.22	0.92	V7	0.07	0.08	0.91
V8	0.20	0.95	0.02	V8	0.08	0.98	−0.12
V9	0.95	0.02	0.02	V9	0.99	−0.11	−0.07
V10	0.79	0.43	0.39	V10	0.74	0.29	0.28

Box problem suggests factors are interpretable

- Thurston/Cattell showed that factors boxes and “plasmodes” recovered interpretable structure
- Overall showed that factor structure of books in his office had 3 factors, but simple structure suggested:
  - text books (fat and big $(X + Y + Z)$)
  - art books (more square, but big on X/Y)
  - novels (thin, $X/Y \approx 5/9$)

Factor analysis of mood

- What is the structure of affect?
- Limited number of variables taken from a larger set

Structure of mood - how not to display data

	AFRAID	AT_EASE	CALM	ENERGETI	HAPPY	LIVELY	SLEEPY	TENSE	TIRED
AFRAID	1.000								
AT_EASE	-0.209	1.000							
CALM	-0.157	0.586	1.000						
ENERGETI	0.019	0.230	0.056	1.000					
HAPPY	-0.070	0.452	0.294	0.595	1.000				
LIVELY	0.018	0.255	0.073	0.778	0.609	1.000			
SLEEPY	0.087	-0.112	0.031	-0.457	-0.264	-0.405	1.000		
TENSE	0.397	-0.337	-0.332	0.088	-0.103	0.084	0.044	1.000	
TIRED	0.082	-0.141	0.012	-0.484	-0.297	-0.439	0.808	0.044	1.000
UNHAPPY	0.350	-0.283	-0.187	-0.185	-0.314	-0.187	0.202	0.360	0.235

Structure of mood: "Alabama need not come first"

	AFRAID	AT_EASE	CALM	ENERGETI	HAPPY	LIVELY	SLEEPY	TENSE	TIRED
AFRAID	1.0								
AT_EASE	-0.2	1.0							
CALM	-0.2	0.6	1.0						
ENERGETI	0.0	0.2	0.1	1.0					
HAPPY	-0.1	0.5	0.3	0.6	1.0				
LIVELY	0.0	0.3	0.1	0.8	0.6	1.0			
SLEEPY	0.1	-0.1	0.0	-0.5	-0.3	-0.4	1.0		
TENSE	0.4	-0.3	-0.3	0.1	-0.1	0.1	0.0	1.0	
TIRED	0.1	-0.1	0.0	-0.5	-0.3	-0.4	0.8	0.0	1.0
UNHAPPY	0.3	-0.3	-0.2	-0.2	-0.3	-0.2	0.2	0.4	0.2

Structure of mood data

	ENERGETI	LIVELY	TIRED	SLEEPY	AFRAID	TENSE	AT_EASE	CALM	HAPPY
ENERGETI	1								
LIVELY	0.8	1							
TIRED	-0.5	-0.4	1						
SLEEPY	-0.5	-0.4	0.8	1					
AFRAID	0	0	0.1	0.1	1				
TENSE	0.1	0.1	0	0	0.4	1			
AT_EASE	0.2	0.3	-0.1	-0.1	-0.2	-0.3	1		
CALM	0.1	0.1	0	0	-0.2	-0.3	0.6	1	
HAPPY	0.6	0.6	-0.3	-0.3	-0.1	-0.1	0.5	0.3	1
UNHAPPY	-0.2	-0.2	0.2	0.2	0.3	0.4	-0.3	-0.2	-0.3

NUMBER OF OBSERVATIONS: 3748

Correlation of mood data possible structure

	ENERGETI	LIVELY	TIRED	SLEEPY	AFRAID	TENSE	AT_EASE	CALM	HAPPY
ENERGETI	1.0								
LIVELY	0.8	1.0							
TIRED	-0.5	-0.4	1.0						
SLEEPY	-0.5	-0.4	0.8	1.0					
AFRAID	0.0	0.0	0.1	0.1	1.0				
TENSE	0.1	0.1	0.0	0.0	0.4	1.0			
AT_EASE	0.2	0.3	-0.1	-0.1	-0.2	-0.3	1.0		
CALM	0.1	0.1	0.0	0.0	-0.2	-0.3	0.6	1.0	
HAPPY	0.6	0.6	-0.3	-0.3	-0.1	-0.1	0.5	0.3	1.0
UNHAPPY	-0.2	-0.2	0.2	0.2	0.3	0.4	-0.3	-0.2	-0.3

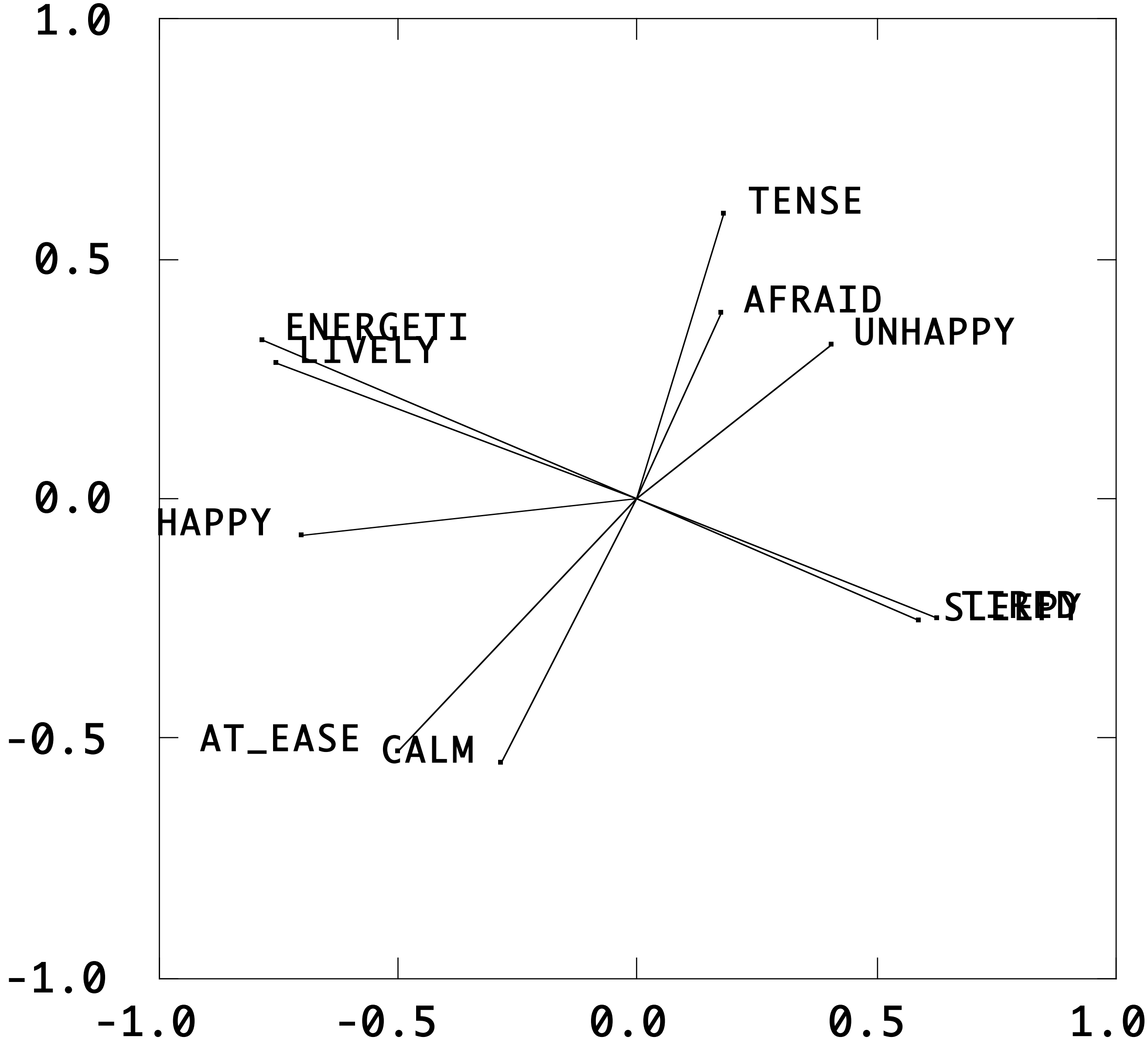
NUMBER OF OBSERVATIONS: 3748

Factor analysis 2 factor solution

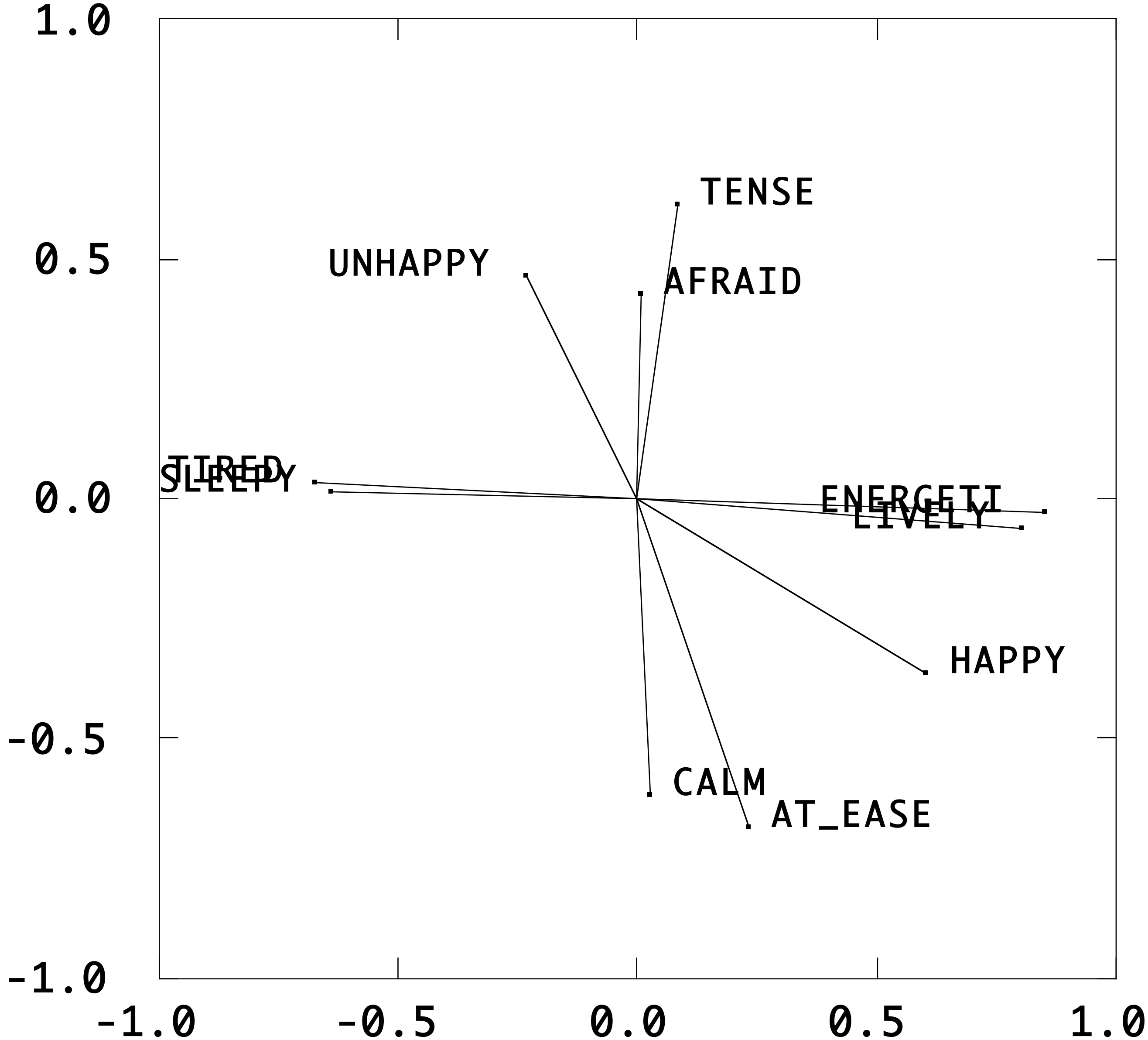
FACTOR PATTERN

	1	2	h2
ENERGETI	-0.8	0.3	0.73
LIVELY	-0.8	0.3	0.73
HAPPY	-0.7	-0.1	0.50
TIRED	0.6	-0.3	0.45
SLEEPY	0.6	-0.3	0.45
TENSE	0.2	0.6	0.40
CALM	-0.3	-0.6	0.45
AT_EASE	-0.5	-0.5	0.50
AFRAID	0.2	0.4	0.20
UNHAPPY	0.4	0.3	0.25
VARIANCE EXP	3.0	1.50	

2 factors of mood



2 factors of mood (rotated)

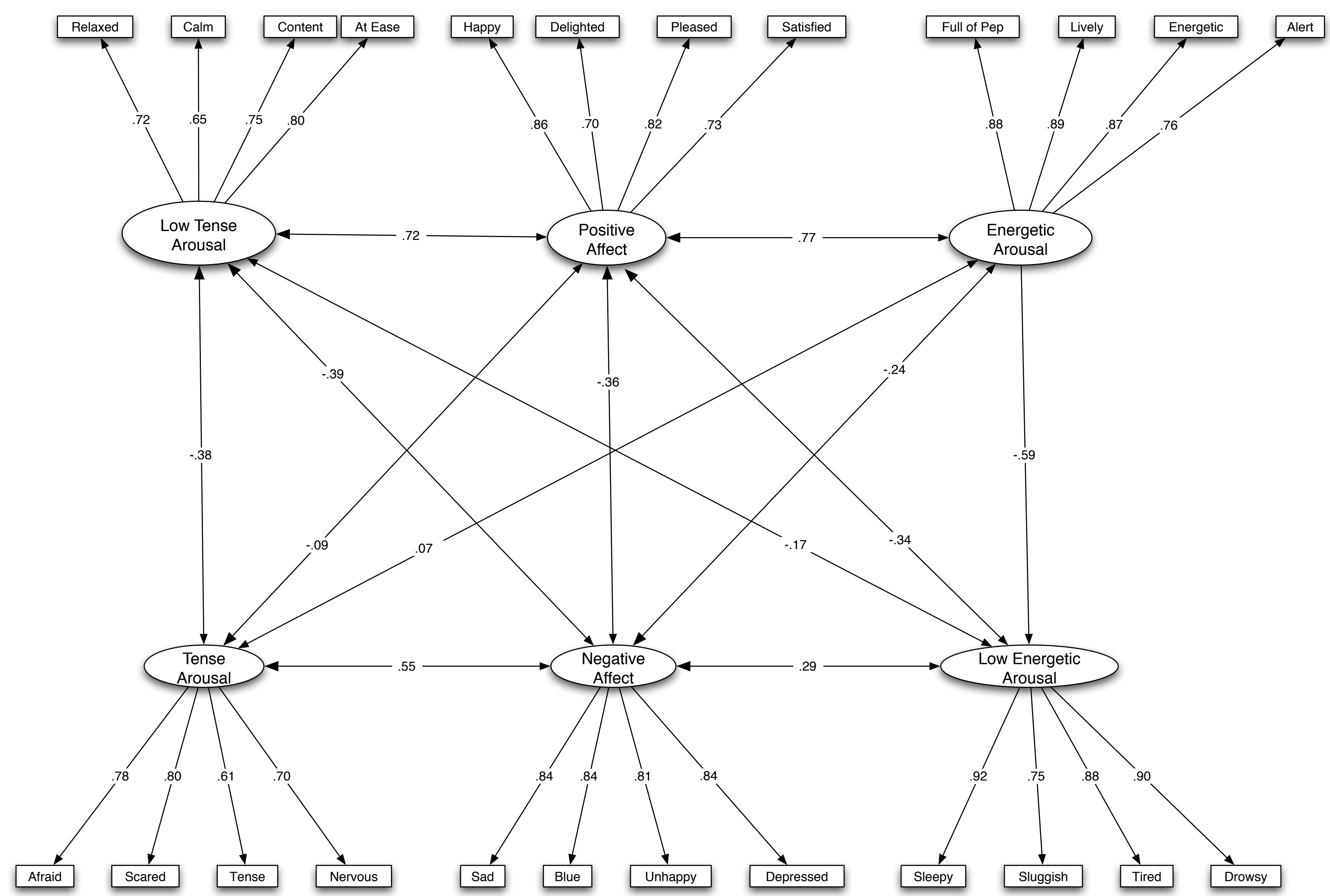


Rotation as orthogonal transformation

	F1	F2	F1'	F2'
ENERGETI	-0.8	0.3	0.8	0.0
LIVELY	-0.8	0.3	0.8	-0.1
HAPPY	-0.7	-0.1	0.6	-0.4
TIRED	0.6	-0.3	-0.7	0.0
SLEEPY	0.6	-0.3	-0.6	0.0
TENSE	0.2	0.6	0.1	0.6
CALM	-0.3	-0.6	0.0	-0.6
AT_EASE	-0.5	-0.5	0.2	-0.7
AFRAID	0.2	0.4	0.0	0.4
UNHAPPY	0.4	0.3	-0.2	0.5
eigen values	3	1.5	2.7	1.8

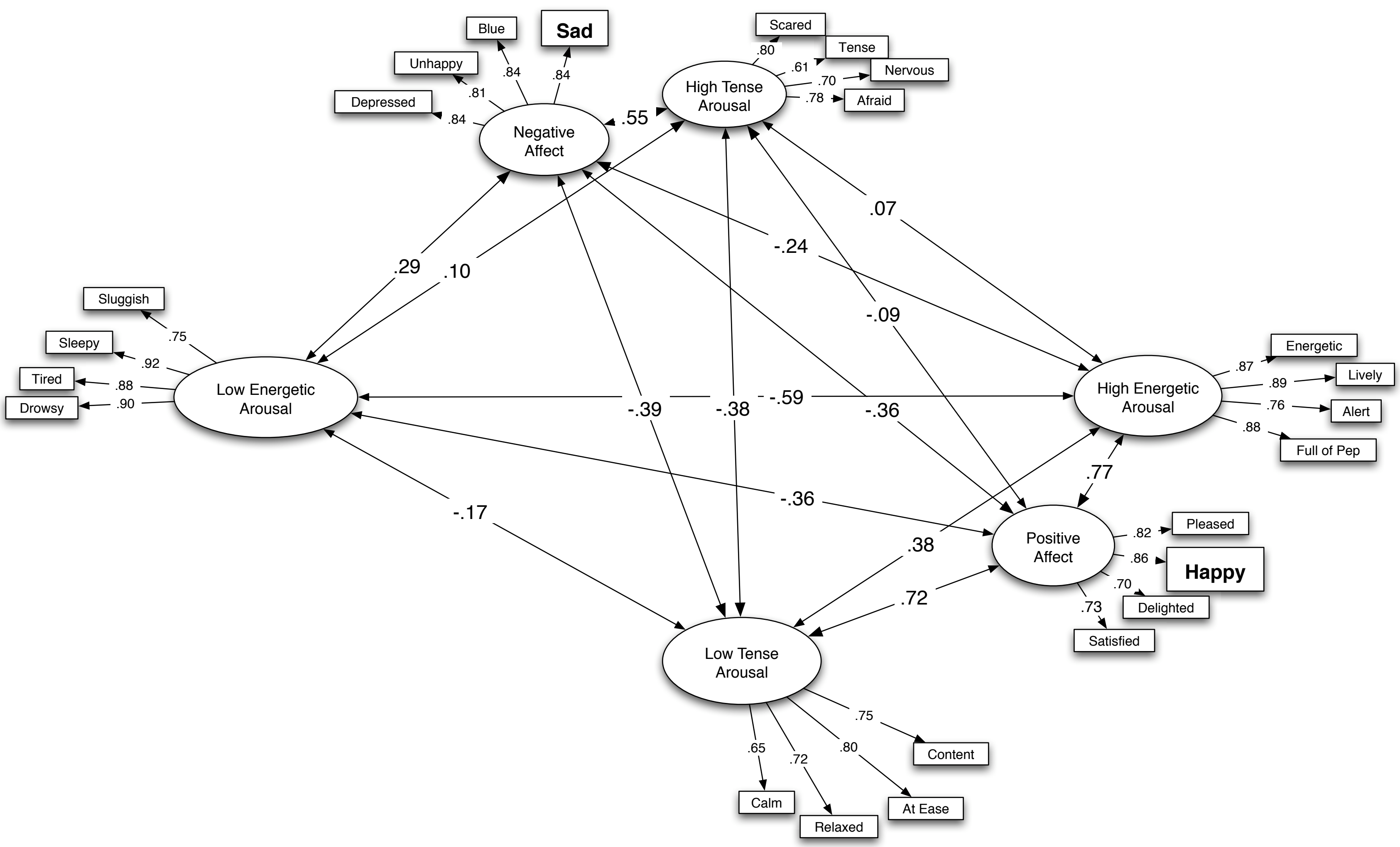
Structure of Affect

- Is happy the opposite of sad?
- Is Positive Affect = - Negative Affect
- What are the dimensions of Affect
- 75 affect words collected over multiple studies for > 3800 subjects



Relaxed
Calm
Content
At Ease
Happy
Delighted
Pleased
Satisfied
Full of Pep
Lively
Energetic
Alert
.72
.65
.75
.80
.86
.70
.82
.73
.88
.89
.87
.76
Low Tense Arousal
Positive Affect
Energetic Arousal
.72
.77
-.39
-.36
-.24
-.38
-.59
-.09
.07
-.17
-.34
Tense Arousal
Negative Affect
Low Energetic Arousal
.55
.29
.78
.80
.61
.70
.84
.84
.81
.84
.92
.75
.88
.90
Afraid
Scared
Tense
Nervous
Sad
Blue
Unhappy
Depressed
Sleepy
Sluggish
Tired
Drowsy

Structure of Affect



Converting Cartesian loadings to polar coordinates

I. Sometimes simpler to organize factor pattern matrix if we think of it in polar coordinates

A. vector length (communality) = $F_1^2 + F_2^2$

B. vector angle (in a two space) =

1. angle = arccosine $(F_1/\mathrm{sqrt}(F_1^2 + F_2^2) * \mathrm{sign}\ (F_2)$

Representative MSQ items (arranged by angular location)

Item	EA-PA	TA-NA	Angle
energetic	**0.8**	0.0	1
elated	**0.7**	0.0	2
excited	**0.8**	0.1	6
anxious	0.2	**0.6**	70
tense	0.1	**0.7**	85
distressed	0.0	**0.8**	93
frustrated	-0.1	**0.8**	98
sad	-0.1	**0.7**	101
irritable	-0.3	**0.6**	114
sleepy	**-0.5**	0.1	164
tired	**-0.5**	0.2	164
inactive	**-0.5**	0.0	177
calm	0.2	**-0.4**	298
relaxed	0.4	**-0.5**	307
at ease	0.4	**-0.5**	312
attentive	**0.7**	0.0	357
enthusiastic	**0.8**	0.0	358
lively	**0.9**	0.0	360

Threats to interpreting correlation and the benefits of structure

- Correlations can be attenuated due to differences in skew (and error)
- Bi polar versus unipolar scales (e.g., of affect)
- How happy do you feel?
  - Not at all A little Somewhat Very
- How sad do you feel?
  - Not at all A little Somewhat Very
- How do you feel?

very sad sad happy very happy

Unipolar scales allow data to speak for themselves

Simulated Example of unipolar scales and the problem of skew

- Consider X and Y as random normal variables
- Let X+ = X if X>0, 0 elsewhere
- Let X- = X if X<0, 0 elsewhere
- Reversed (X+) = - X+
- Similarly for Y
- Examine the correlational structure
- Note that although X and -X correlate -1.0, X+ and X- correlate only -.43 and that X+ correlates with X+Y+ .66

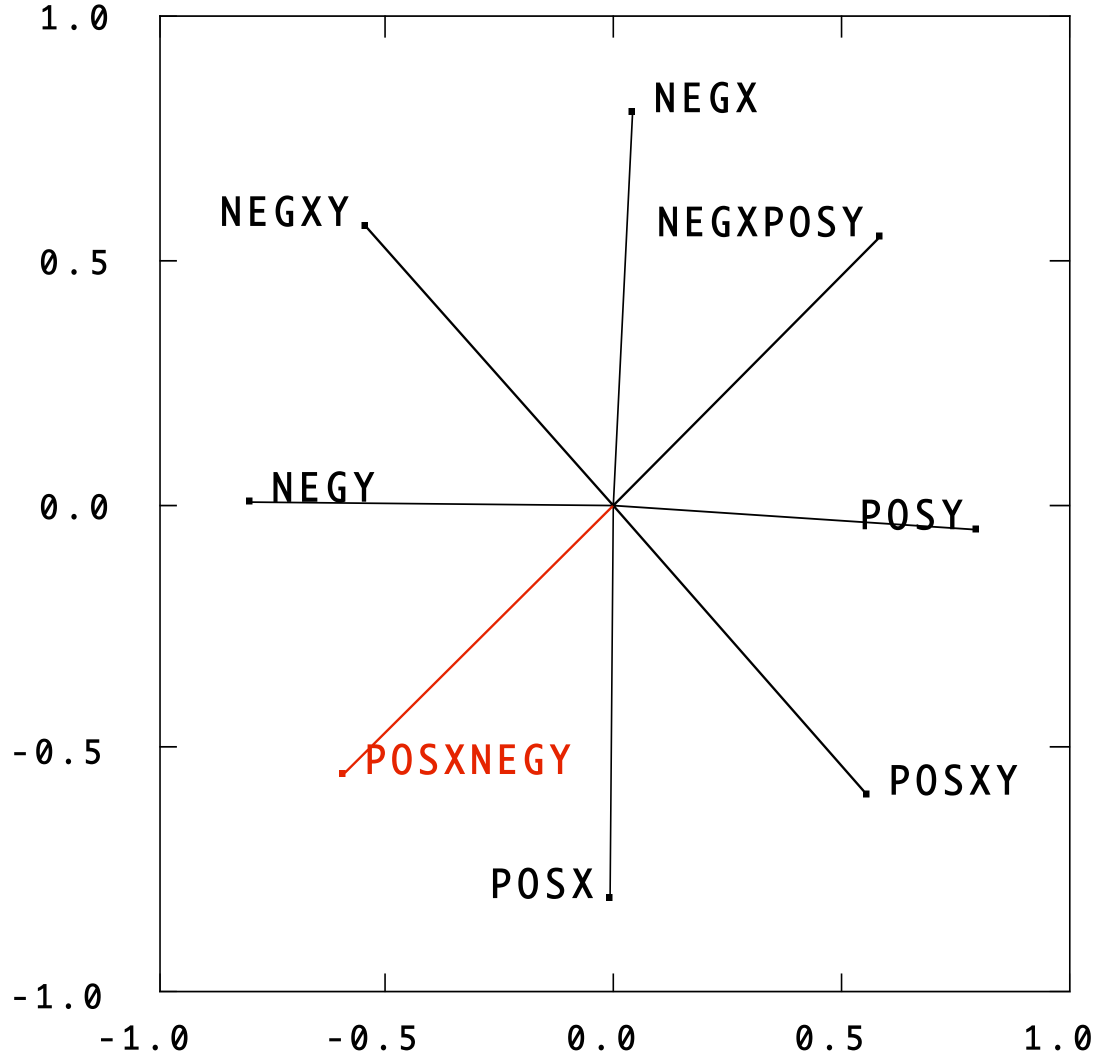
Determining Structure: zeros and skew

	X+	X-	Y+	Y-	X+Y+	X-Y-	X+Y-	X-Y+
X+	1.00							
X-	-0.47	1.00						
Y+	0.03	-0.01	1.00					
Y-	0.00	-0.03	-0.46	1.00				
X+Y+	0.65	-0.39	0.66	-0.39	1.00			
X-Y-	-0.40	0.63	-0.40	0.63	-0.46	1.00		
X+Y-	0.63	-0.40	-0.39	0.66	0.00	0.02	1.00	
X-Y+	-0.39	0.64	0.63	-0.40	0.00	0.00	-0.47	1.00

Factor analysis shows structure

	1	2	
POSX	-0.01	-0.81	271
NEGX	0.04	0.80	87
POSY	0.80	-0.05	356
NEGY	-0.80	0.00	180
POSXY	0.56	-0.59	314
NEGXY	-0.55	0.57	136
POSXN	-0.59	-0.55	227
NEGXP	0.58	0.55	43

Structural representation



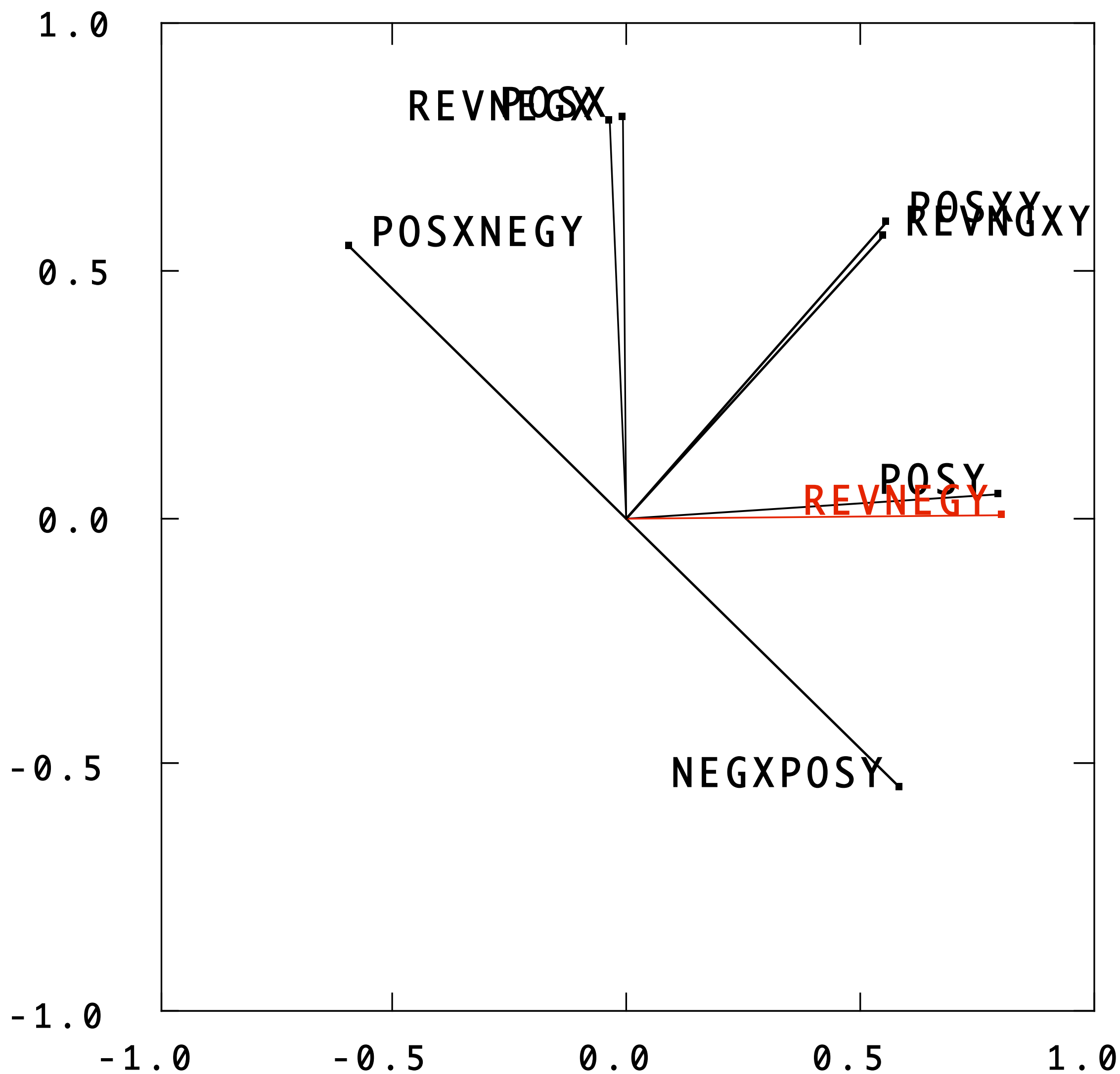
Skew and zeros: determining structure

	Y+	X+Y+	X+	X+Y-	r(Y-)	r(X-Y-)	r(X-)	X-Y+
Y	1.00							
X+Y+	0.66	1.00						
X+	0.03	0.65	1.00					
X+Y-	-0.39	0.00	0.63	1.00				
r(Y-)	0.46	0.39	0.00	-0.66	1.00			
r(X-Y-)	0.40	0.46	0.40	-0.02	0.63	1.00		
r(X-)	0.01	0.39	0.47	0.40	-0.03	0.63	1.00	
X-Y+	0.63	0.00	-0.39	-0.47	0.40	0.00	-0.64	1.00

Factor analysis shows structure

	1	2	angle
POSX	-0.01	0.81	89
POSY	0.80	0.05	4
POSXY	0.56	0.59	46
POSXN	-0.59	0.55	137
NEGXP	0.58	-0.55	313
REVNE	-0.04	0.80	87
REVNE	0.80	0.00	90
REVNG	0.55	0.57	46

Factor analysis shows structure



Hyperplanes and Zeros: Defining variables by what they are not

- Tendency to interpret variables in terms of their patterns of high correlations
- But large correlations may be attenuated by skew or error
- Correlation of .7 is an angle of 45 degrees => lots of room for misinterpretation!
- Zero correlations reflect what a variable is not
  - Zeros provide definition by discriminant validity

Other rotations/extractions

- Extraction
  - Maximum Likelihood
  - Minimum Residual
  - Centroid
  - Principal Components (not factors!, not latent!)
- Transformations (not necessarily orthogonal)
  - Procrustes (force to a solution)
  - Promax

Factors and Factor scores

- The indeterminacy of factor scores
  - at the data level the model is not defined and factor scores are estimated, not found
  - as number of factors increases and sample size decreases, indeterminacy increases

- components can be found directly from the data (which appeals to some)

Hierarchical models

- Ability tests typically analyzed in terms of hierarchical models
  - g -
    - gf, gc, ...
    - matrices, reasoning, working memory, ...
- Hierarchical models becoming more accepted in non-cognitive domains
  - N
    - Anxiety, Depression, Anger
    - test anxiety, public speaking, ...

Hierarchical models

- Allow first order factors to correlate
  - find correlations of these factors
  - factor the correlation matrix to get second order factors,
  - some (i.e., Cattell) would continue on up to 3rd and fourth level factors
- In item/scale construction, technique is to form homogeneous item composites and then factor them