Path diagrams and matrix multiplication

A picture is worth a thousand words

Representing the data

I. Correlation, covariance, and regression models may be represented several different ways:

   A. as correlation/covariance matrices - the data to be fit

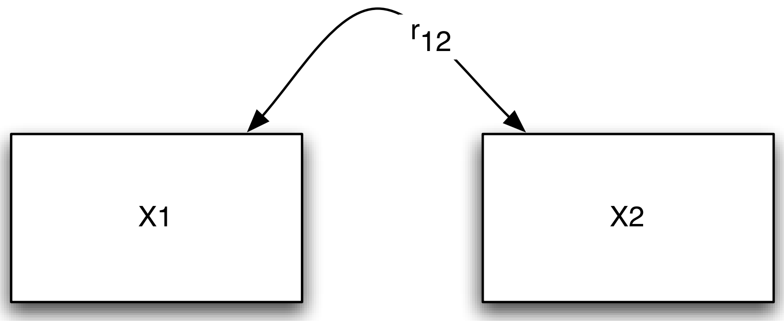
   B. as Venn diagrams (for 2-3 variables)

   C. as simultaneous equations

   D. as path models

   E. as matrix equations

Correlation or regression



	X1	X2
X1	1	r_{12}
X2	r_{12}	1

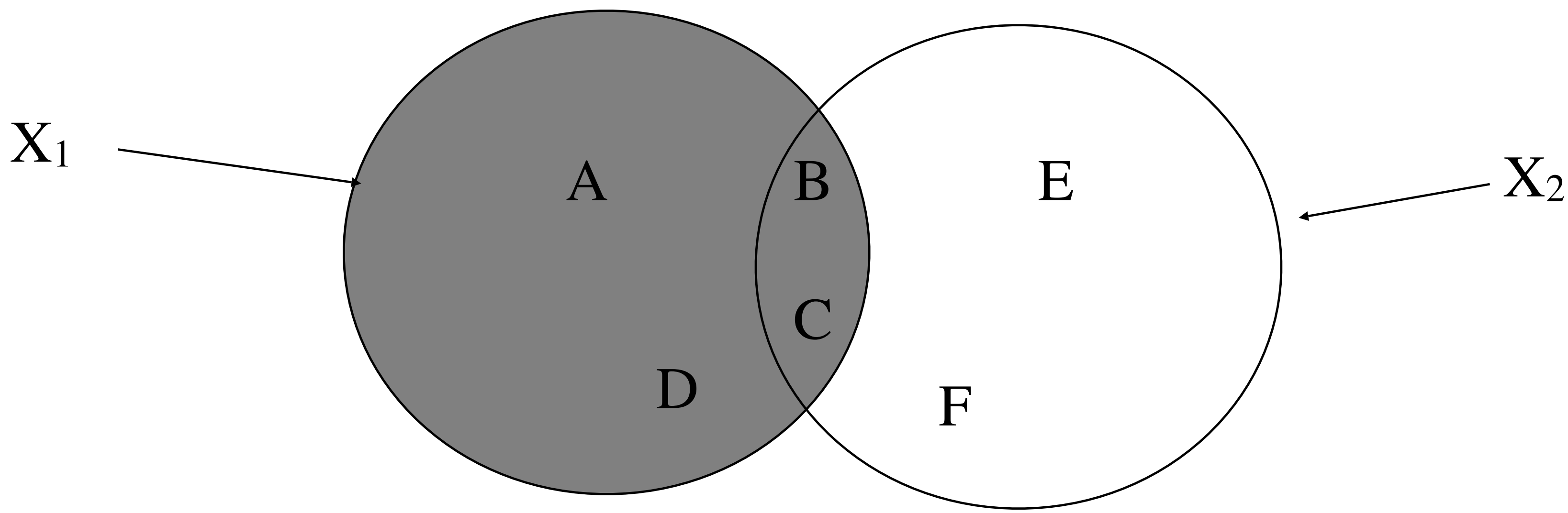
Correlation: The Venn diagram approach

I. Represent the variance as total area

II. Represent covariance as overlap

III. Possible to decompose Variance into Unique and shared variances

Variance, Covariance and Correlation



$V_1 = A + B + C + D$ $\quad$ $C_{12} = B + C$ $\quad$ $V_2 = E + B + C + F$

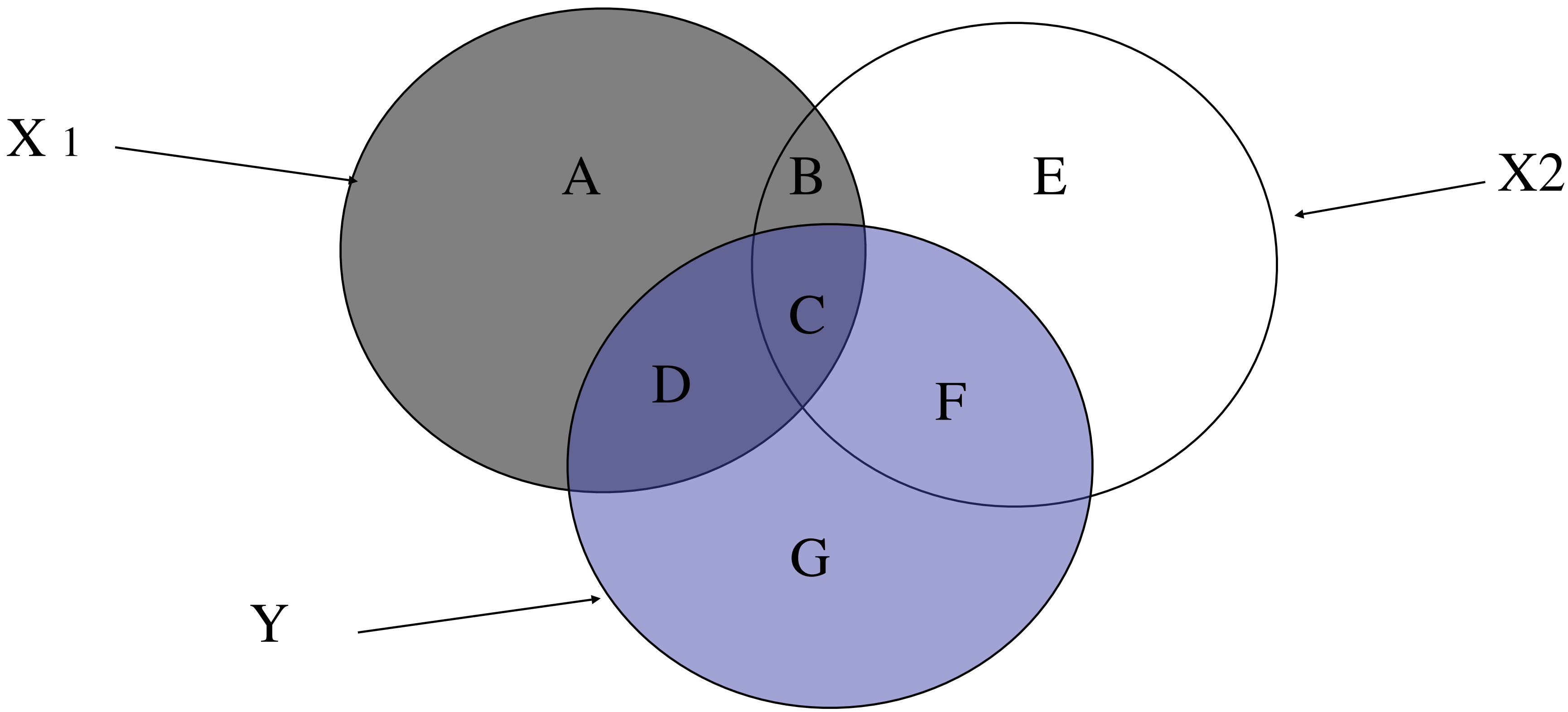
$r = C_{12}/\mathrm{sqrt}(V_1 V_2)$

$V_{1.2} = A + D$ $\quad$ $V_{2.1} = E + F$

$V_{1.2} = V_1(1-r^2)$ $\quad$ $V_{2.1} = V_2(1-r^2)$

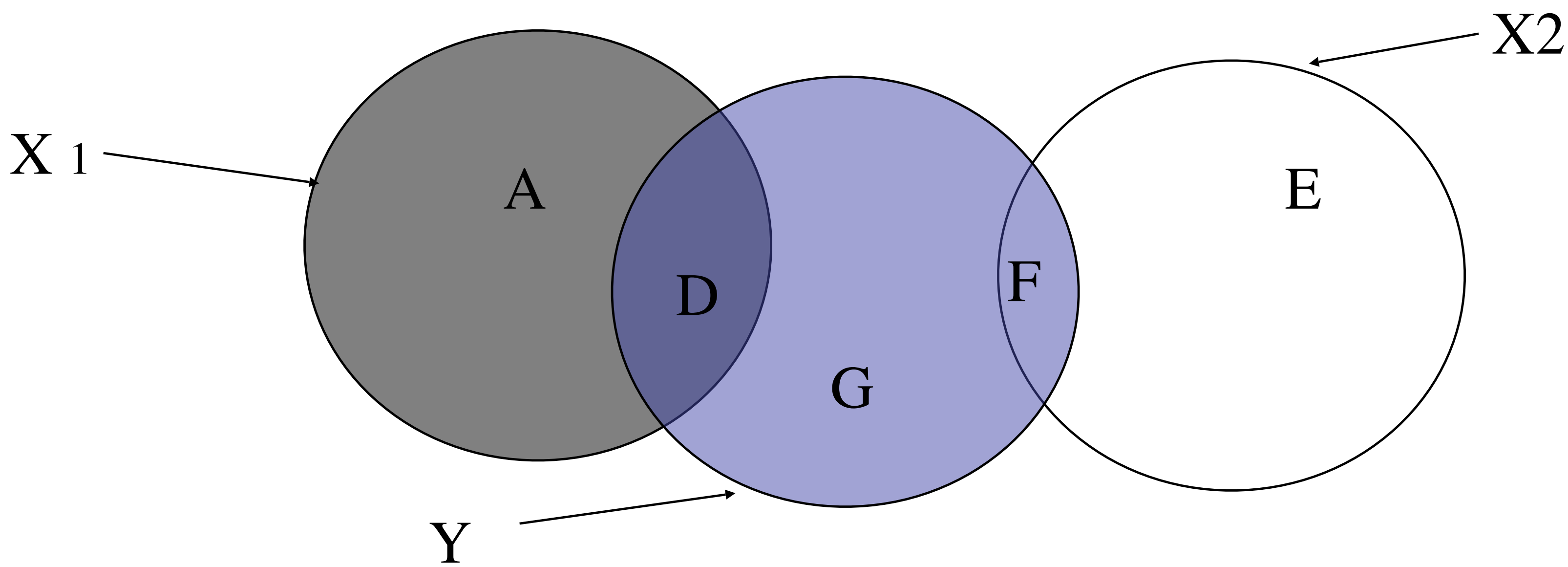
Partial and Multiple Correlation

The conceptual problem



Multiple Correlation

Independent Predictors



$V_1 = A + B + C + D$

$V_2 = E + B + C + F$

$V_Y = D + C + F + G$

$V_{1.2} = A + D$

$C_{12} = B + C$

$C_{1Y} = C + D$

$C_{2Y} = C + F$

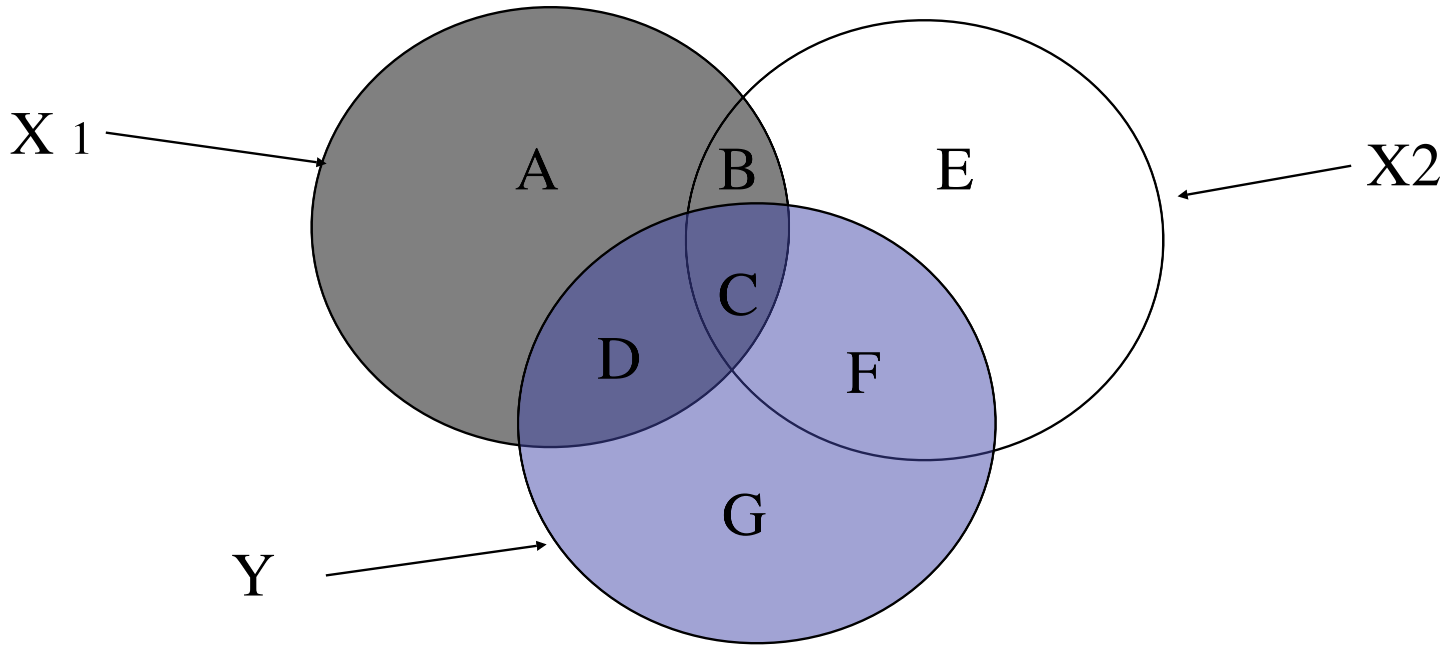
$V_{2.}1 = E + F$

$C_{1Y.2} = D$

$C_{2Y.1} = F$

$C_{(12)Y} = D + C + F$

Partial and Multiple Correlation



$V_1 = A + B + C + D$

$V_2 = E + B + C + F$

$V_Y = D + C + F + G$

$V_{1.2} = A + D$

$C_{12} = B + C$

$C_{1Y} = C + D$

$C_{2Y} = C + F$

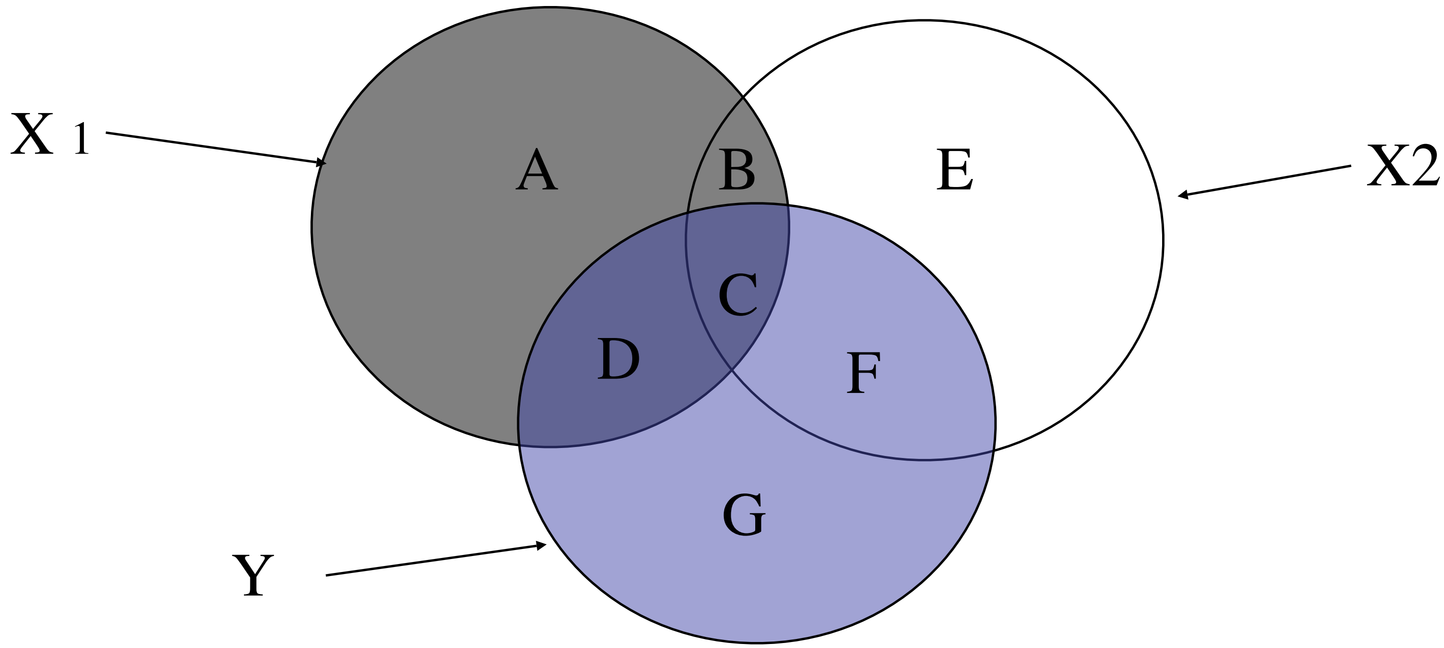
$V_{2.1} = E + F$

$C_{1Y.2} = D$

$C_{2Y.1} = F$

$C_{(12)Y} = D + C + F$

Partial and Multiple Correlation: Partial Correlations



$V_1 = A + B + C + D$

$V_2 = E + B + C + F$

$V_Y = D + C + F + G$

$V_{1.2} = A + D$

$C_{12} = B + C$

$C_{1Y} = C + D$

$C_{2Y} = C + F$

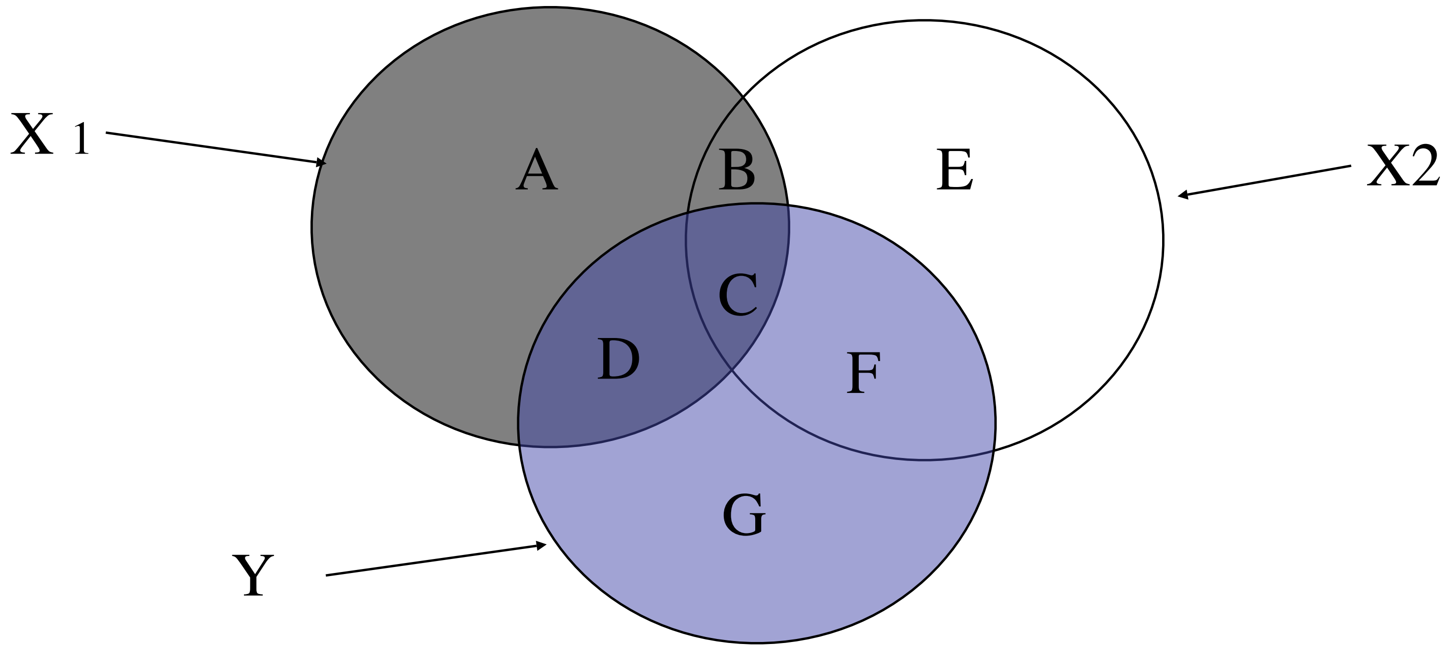
$V_{2.1} = E + F$

$C_{1Y.2} = D$

$C_{2Y.1} = F$

$$r_{1Y.2} = \frac{(r_{1y} - r_{12} * r_{2Y})}{\mathrm{sqrt}((1 - r_{12}^2) * (1 - r_{y2}^2))}$$

Partial and Multiple Correlation: Multiple Correlation-correlated predictors



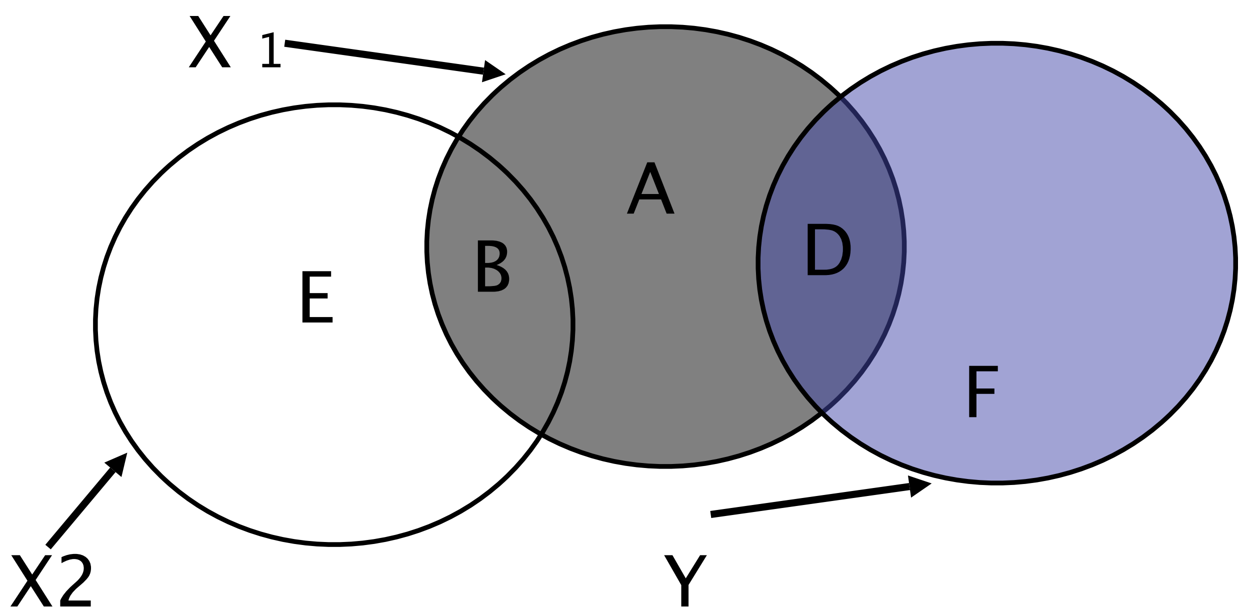
$Y = b_1X_1 + b_2X_2$

$b_1 = (r_{x1y} - r_{12}*r_{2y})/(1-r_{12}^2)$

$b_2 = (r_{x2y} - r_{12}*r_{1y})/(1-r_{12}^2)$

$R^2 = b_1r_1+b_2r_2$

Multiple Correlation:



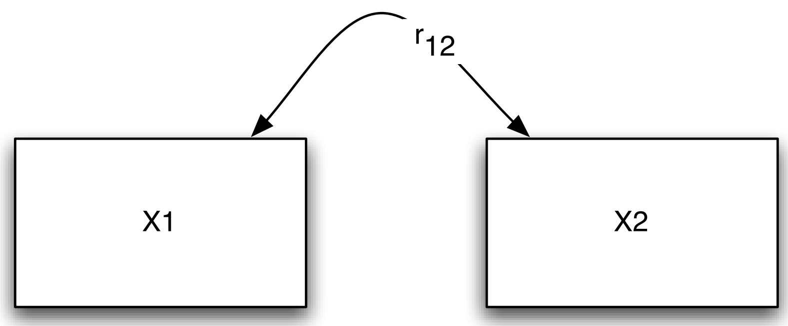
$V_1 = A + B + C + D$ $C_{12} = B + C$ $C_{1Y.2} = D$

$V_2 = E + B + C + F$ $C_{1Y} = C + D$ $C_{2Y.1} = F$

$V_Y = D + C + F + G$ $C_{2Y} = C + F$ $C_{(12)Y} = D + C + F$

$V_{1.2} = A + D$ $V_{2.1} = E + F$

Correlation or regression



	X1	X2
X1	1	r_{12}
X2	r_{12}	1

Correlation does not imply direction of influence
Regression typically taken as direction of influence

Correlation/Regression and causality

I. Correlation is a standardized covariance

   A. Correlation = $\text{Covariance}_{xy}/\text{sqrt}(V_x V_y)$

   B. unit free measure of relationship

II. Regression is $\text{Covariance}_{xy}/\text{Variance}_x$

   A. Slope (ß) is in units of change in Y as f(change in X)

   B. Does not necessarily imply causality but is frequently taken that way

Correlation and causality

I. Shoe size predicts verbal ability among high school students

II. Salaries of Methodist ministers in Evanston predicts price of rum in Puerto Rico

III. Yellowed fingers and bad breath predict lung cancer

IV. CO_2 levels at Mauna Loa predict global warming

V. Years of education predict mortality rates

Structural Equation Modeling sometimes called Causal Modeling

I. By summarizing patterns of relationships with directional arrows we can fall for the trap of interpreting fitting the data as explaining the data.

II. Causal reasoning and interpretation is beyond the scope of what simple model fitting programs can do.

Correlation and Regression as path models or matrix models

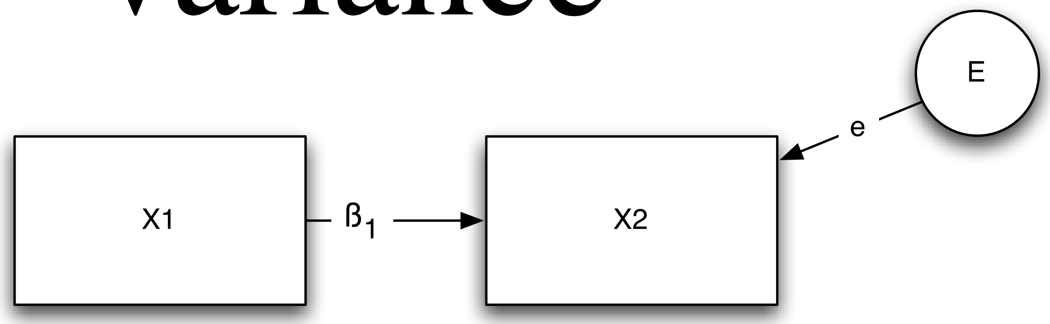
I. Path notation shows Pattern of relationships

   A. path arithmetic

      1. no loops
      2. one curved arrow/path
      3. no forward and then back

II. Matrix notation of paths can show Pattern, Structure, and represent data (and allow for calculation)

Regression: Modeling the variance



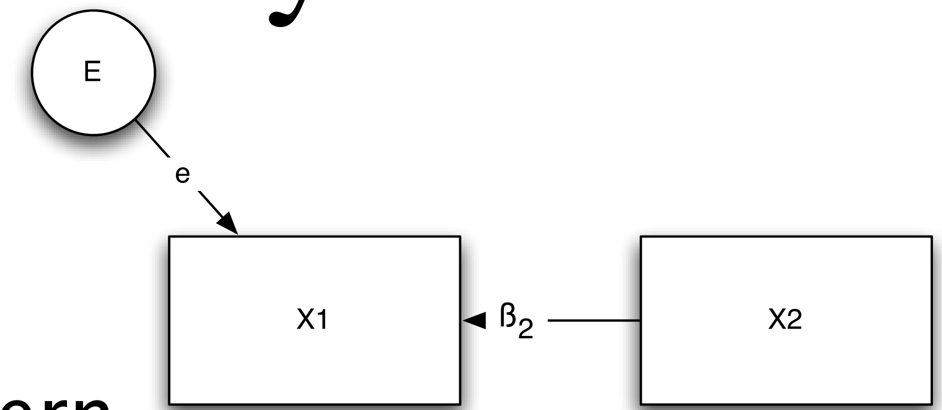
Pattern

	X1	E
X1	1	0
X2	ß	e

PP'

	X1	X2
X1	1	ß
X2	ß	$ß^2+e^2$

Correlation or regression: which way is the direction



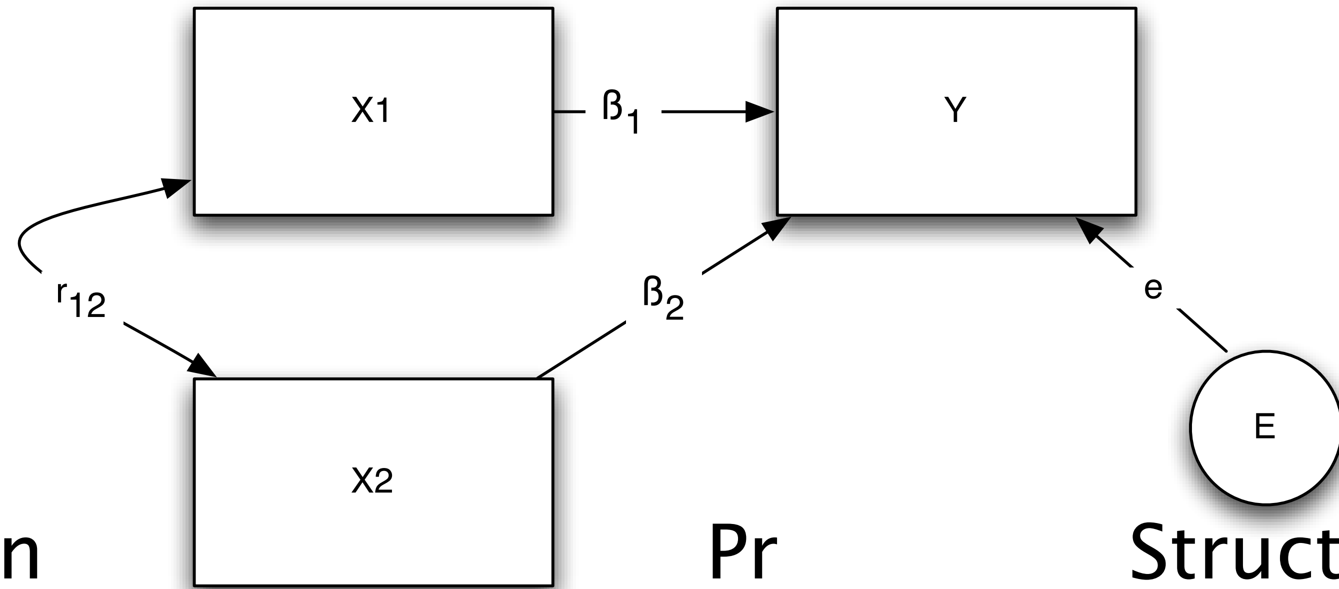
Pattern

	X2	E
X1	ß	0
X2	1	e

PP'

	X1	X2
X1	$ß^2+e^2$	ß
X2	ß	1

Multiple regression



Pattern

	X_1	X_2	E
X_1	1	0	0
X_2	0	1	0
Y	$ß_1$	$ß_2$	e

Pr

	X_1	X_2	E
X_1	1	r_1	0
X_2	r_1	1	0
E	0	0	1

Structure

	X_1	X_2	Y
X_1	1	r_{12}	0
X_2	r_{12}	1	0
Y	$ß_1+ß_2r$	$ß_1r+ß_2$	e

Multiple Regression as a set of simultaneous equations

$$\left\{ \begin{array}{ccc} r_{x1x1} & r_{x1x2} & r_{x1y} \\ r_{x1x2} & r_{x2x2} & r_{x2y} \\ r_{x1y} & r_{x2y} & r_{yy} \end{array} \right\}$$

$$\left\{ \begin{array}{c} r_{x1x1}\beta_1 + r_{x1x2}\beta_2 = r_{x1y} \\ r_{x1x2}\beta_1 + r_{x2x2}\beta_2 = r_{x2y} \end{array} \right\}.$$

$$\left\{ \begin{array}{c} \beta_1 = (r_{x1y}r_{x2x2} - r_{x1x2}r_{x2y})/(r_{x1x1}r_{x2x2} - r_{x1x2}^2) \\ \beta_2 = (r_{x2y}r_{x1x1} - r_{x1x2}r_{x1y})/(r_{x1x1}r_{x2x2} - r_{x1x2}^2) \end{array} \right\}$$

Matrix representation

$$(\beta_1 \beta_2) \begin{pmatrix} r_{x1x1} & r_{x1x2} \\ r_{x1x2} & r_{x2x2} \end{pmatrix} = (r_{x1y} \quad r_{x2x2})$$

$$\beta = (\beta_1 \beta_2),\ \mathrm{R} = \begin{pmatrix} r_{x1x1} & r_{x1x2} \\ r_{x1x2} & r_{x2x2} \end{pmatrix} \text{ and } r_{xy} = (r_{x1y} \quad r_{x2x2})$$

$$\beta R = r_{xy}$$

$$\beta = \beta R R^{-1} = r_{xy} R^{-1}$$

Finding the inverse

$$R = IR$$

$$\begin{pmatrix} r_{x1x1} & r_{x1x2} \\ r_{x1x2} & r_{x2x2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{x1x1} & r_{x1x2} \\ r_{x1x2} & r_{x2x2} \end{pmatrix}$$

$$T_1 = \begin{pmatrix} \frac{1}{r_{11}} & 0 \\ 0 & \frac{1}{r_{22}} \end{pmatrix}$$

The inverse of a matrix

$$T_1 R = T_1 I R$$

$$\begin{pmatrix} 1 & \frac{r_{12}}{r_{11}} \\ \frac{r_{12}}{r_{22}} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r_{11}} & 0 \\ 0 & \frac{1}{r_{22}} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}$$

...

$$T_3 T_2 T_1 R = I = R^{-1} R$$

$$T_3 T_2 T_1 I = R^{-1}$$

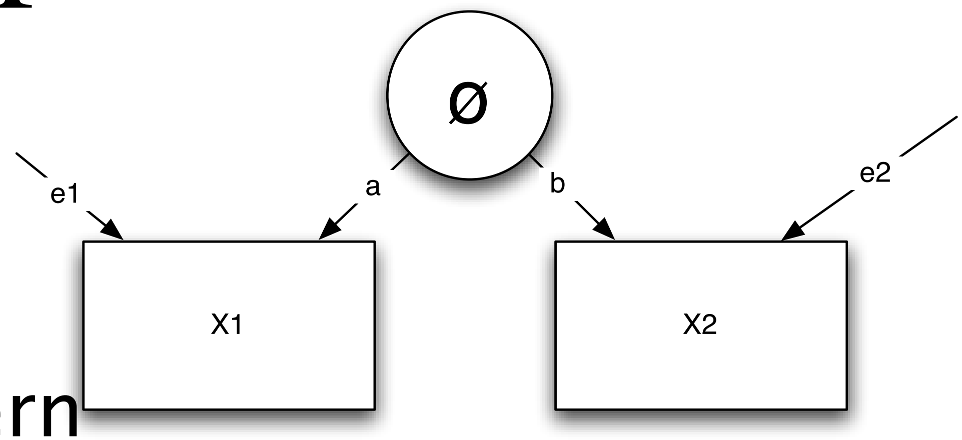
The inverse of a 2 x 2

```
> R2
     x1   x2
x1 1.00 0.56
x2 0.56 1.00
> round(solve(R2),2)
      x1    x2
x1  1.46 -0.82
x2 -0.82  1.46
```

The inverse of a 3 x 3

```
> R
     x1   x2   x3
x1 1.00 0.56 0.48
x2 0.56 1.00 0.42
x3 0.48 0.42 1.00
> round(solve(R),2)
      x1    x2    x3
x1  1.63 -0.71 -0.48
x2 -0.71  1.52 -0.30
x3 -0.48 -0.30  1.36
```

Parallel tests: an under specified model



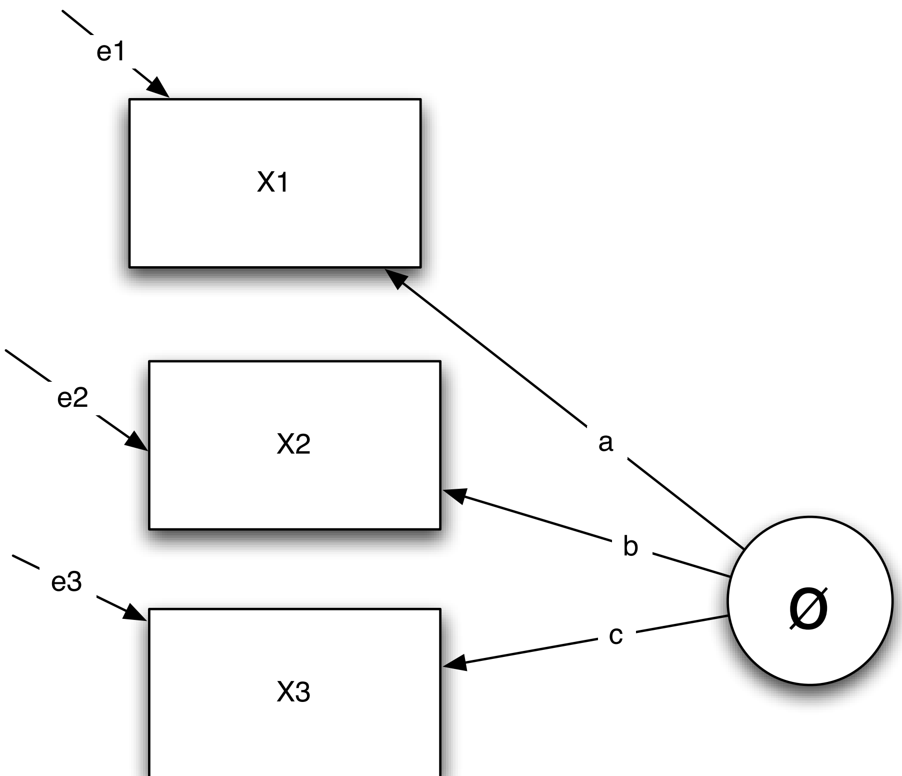
Pattern

	ø	E_1	E_2
X1	a	e_1	0
X2	b	0	e_2

PP'

	X1	X2
X1	$a^2+e_1^2$	ab
X2	ab	$b^2+e_2^2$

Tau equivalent tests: an exactly specified model



Pattern

	ø	E_1	E_2	E_3
X1	a	e_1	0	0
X2	b	0	e_2	0
X3	c	0	0	e_3

Tau equivalent tests

Pattern

	ø	E_1	E_2	E_3
X1	a	e_1	0	0
X2	b	0	e_2	0
X3	c	0	0	e_3

PP’

	X_1	X_2	X_3
X_1	$a^2+e_1{}^2$	bc	ac
X_2	ab	$b^2+e_2{}^2$	bc
X_3	ac	bc	$c^2+e_3{}^2$

tau equivalent tests

Pattern

	ø	E_1	E_2	E_3
X1	.9	.2	0	0
X2	.8	0	.4	0
X3	.6	0	0	.6

PP'

	X_1	X_2	X_3
X_1	$a^2+e_1^2$	bc	ac
X_2	ab	$b^2+e_2^2$	bc
X_3	ac	bc	$c^2+e_3^2$

Tau equivalent model

```
P <- matrix(c(.9,.2,0,0,
+             .8,0,.4,0,
+             .6,0,0,.6),nrow=3,byrow=TRUE)
> rownames(P) <- c("X1","X2","X3")
> colnames(P) <- c("theta", "E1","E2","E3")
> P
```

Pattern

	theta	E1	E2	E3
X1	0.9	0.2	0.0	0.0
X2	0.8	0.0	0.4	0.0
X3	0.6	0.0	0.0	0.6

```
> R <- P %*% t(P)
R
```

PP'

	X1	X2	X3
X1	0.85	0.72	0.54
X2	0.72	0.80	0.48
X3	0.54	0.48	0.72

A second tau equivalent model

```
 P <- matrix(c(.9,.44,0,0,
+              .8,0,.6,0,
+              .6,0,0,.8),nrow=3,byrow=TRUE)
>              rownames(P) <- c("X1","X2","X3")
> colnames(P) <- c("theta", "E1","E2","E3")
> P
```

Pattern

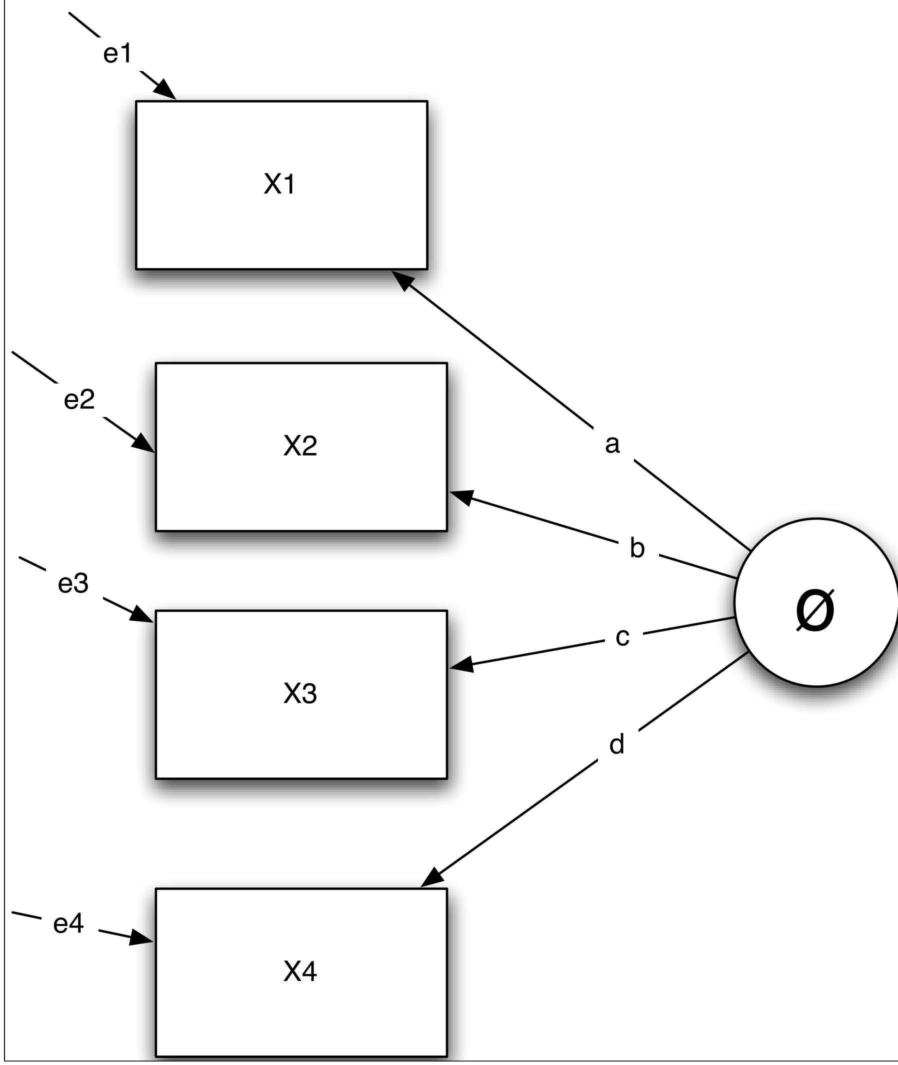
	theta	E1	E2	E3
X1	0.9	0.44	0.0	0.0
X2	0.8	0.00	0.6	0.0
X3	0.6	0.00	0.0	0.8

```
> R <- P %*% t(P)
```

PP’

	X1	X2	X3
X1	1.00	0.72	0.54
X2	0.72	1.00	0.48
X3	0.54	0.48	1.00

Congeneric models: overspecified and thus testable



Pattern

	ø	E_1	E_2	E_3	E_4
X_1	a	e_1	0	0	0
X_2	b	0	e_2	0	0
X_3	c	0	0	e_3	0
X_4	d	0	0	0	e_4

A congeneric model

P

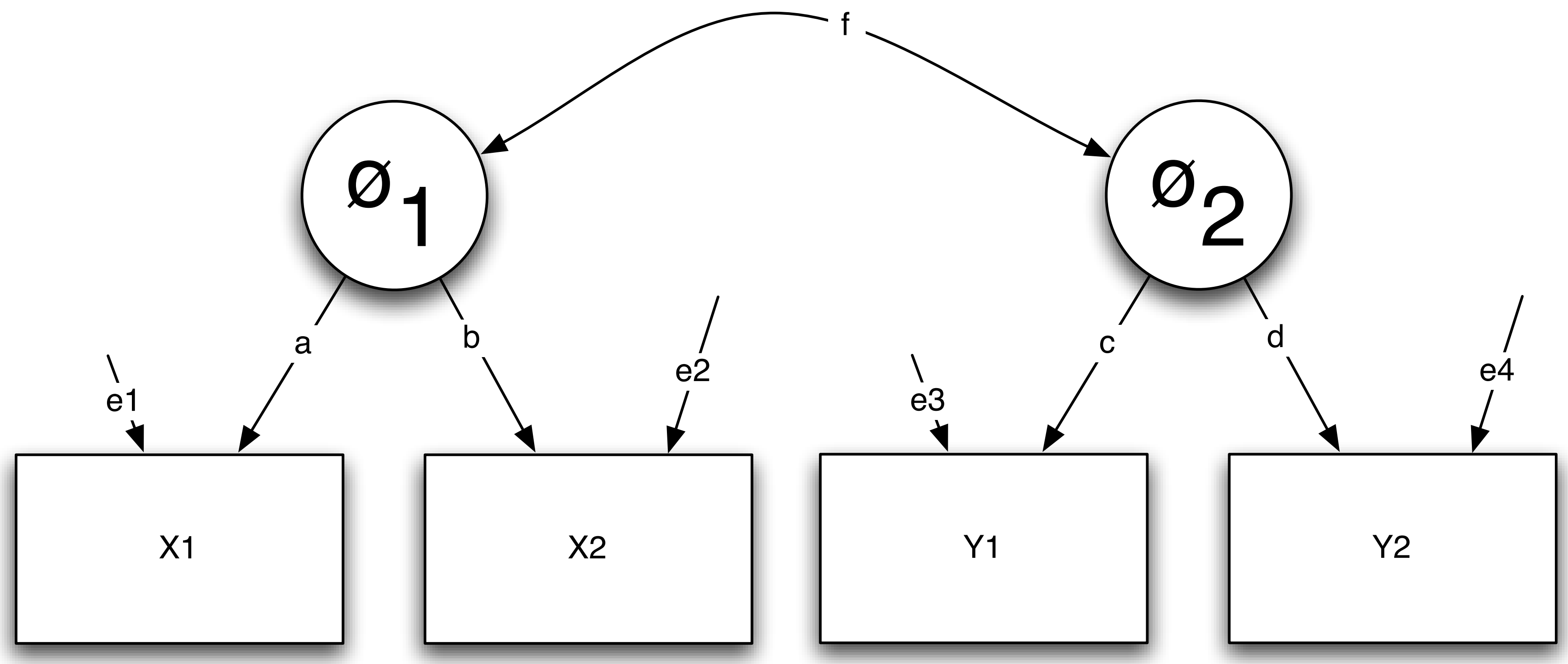
	theta	E1	E2	E3	E4
X1	0.9	0.44	0.0	0.00	0.0
X2	0.8	0.00	0.6	0.00	0.0
X3	0.7	0.00	0.0	0.71	0.0
X4	0.6	0.00	0.0	0.00	0.8

```
> R <- P %*% t(P)
```

PP’

	X1	X2	X3	X4
X1	1.00	0.72	0.63	0.54
X2	0.72	1.00	0.56	0.48
X3	0.63	0.56	0.99	0.42
X4	0.54	0.48	0.42	1.00

Basic path model



Pattern * Correl = Structure

Pattern

	$ø_1$	$ø_2$
X_1	a	0
X_2	b	0
X_3	0	c
X_4	0	d
$ø_1$	1	0
$ø_2$	0	1

Correlation

	$ø_1$	$ø_2$
$ø_1$	1	f
$ø_2$	f	1

Structure

	$ø_1$	$ø_2$
X_1	a	af
X_2	b	bf
X_3	cf	c
X_4	df	d
$ø_1$	1	f
$ø_2$	f	1

Structure = Pattern x correlation

P1

	theta1	theta2
X1	0.8	0.0
X2	0.7	0.0
X3	0.0	0.8
X4	0.0	0.9

Pr

	theta1	theta2
theta1	1.0	0.4
theta2	0.4	1.0

S = P1 Pr

S

	theta1	theta2
X1	0.80	0.32
X2	0.70	0.28
X3	0.32	0.80
X4	0.36	0.90

Model = Pattern x correl x P'

P1

	theta1	theta2
X1	0.8	0.0
X2	0.7	0.0
X3	0.0	0.8
X4	0.0	0.9

Pr

	theta1	theta2
theta1	1.0	0.4
theta2	0.4	1.0

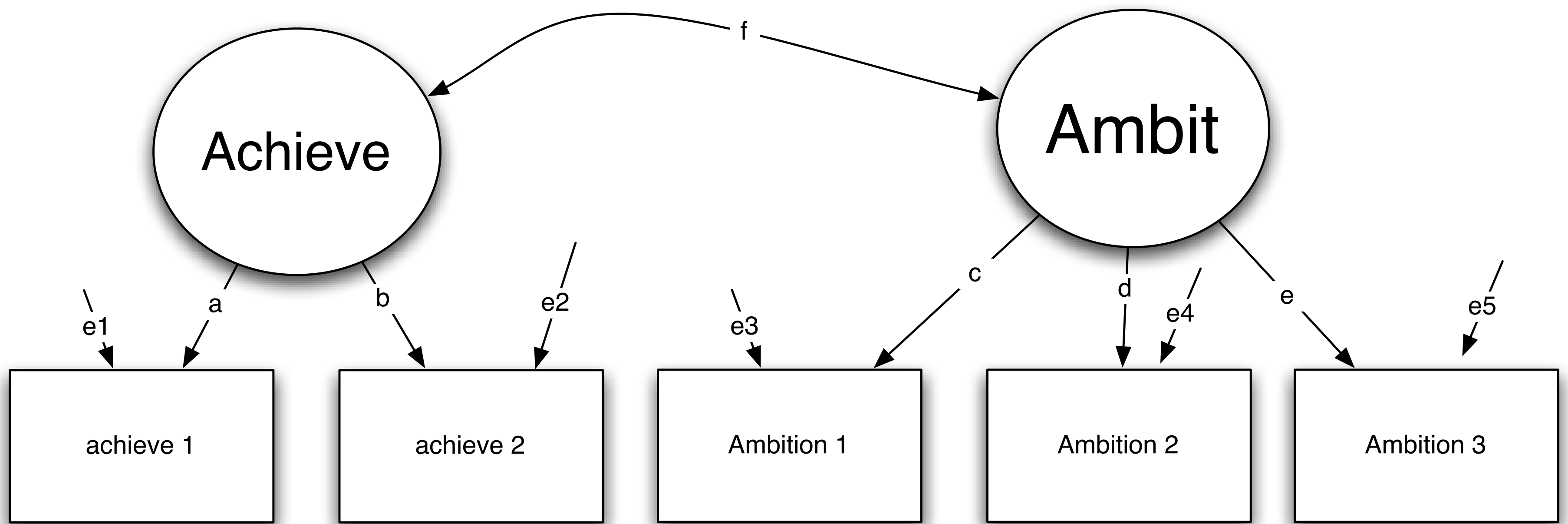
model = P1 Pr P1'

	X1	X2	X3	X4
X1	0.64	0.56	0.26	0.29
X2	0.56	0.49	0.22	0.25
X3	0.26	0.22	0.64	0.72
X4	0.29	0.25	0.72	0.81

A SEM
Loehlin problem 2.5

	Ach1	Ach2	Amb1	Amb2	Amb3
Ach1	1.0	0.6	0.3	0.2	0.2
Ach2	0.6	1.0	0.2	0.3	0.1
Amb1	0.3	0.2	1.0	0.7	0.6
Amb2	0.2	0.3	0.7	1.0	0.5
Amb3	0.2	0.1	0.6	0.5	1.0

A possible path model



SEM analysis

Parameter Estimates

	Estimate	Std Error	z value	Pr(>|z|)	
a	0.920	0.0924	9.966	0.00e+00	Amb1 <--- Ambit
b	0.761	0.0955	7.974	1.55e-15	Amb2 <--- Ambit
c	0.652	0.0965	6.753	1.45e-11	Amb3 <--- Ambit
d	0.879	0.1762	4.986	6.16e-07	Ach1 <--- Achieve
e	0.683	0.1509	4.525	6.03e-06	Ach2 <--- Achieve
f	0.356	0.1138	3.127	1.76e-03	Achieve <--> Ambit
u	0.153	0.0982	1.557	1.20e-01	Amb1 <--> Amb1
v	0.420	0.0898	4.679	2.88e-06	Amb2 <--> Amb2
w	0.575	0.0949	6.061	1.35e-09	Amb3 <--> Amb3
x	0.228	0.2791	0.816	4.15e-01	Ach1 <--> Ach1
y	0.534	0.1837	2.905	3.67e-03	Ach2 <--> Ach2

Loehlin Prob 2.5 (cont)

Pattern

	Ambit	Achiev
Ach1	0.00	0.88
Ach2	0.00	0.68
Amb1	0.92	0.00
Amb2	0.76	0.00
Amb3	0.65	0.00

Pr1

	Ambit	Achiev
Ambit	1.000	0.356
Achiev	0.356	1.000

round(Struct,2)

	Ambit	Achiev
Ach1	0.31	0.88
Ach2	0.24	0.68
Amb1	0.92	0.33
Amb2	0.76	0.27
Amb3	0.65	0.23

Loehlin 2.5 continued

```
> Pattern
      Ambit Achiev
Ach1   0.00   0.88
Ach2   0.00   0.68
Amb1   0.92   0.00
Amb2   0.76   0.00
Amb3   0.65   0.00
```

```
Pr1
       Ambit Achiev
Ambit  1.000  0.356
Achiev 0.356  1.000
```

```
round(Model,2)
     Ach1 Ach2 Amb1 Amb2 Amb3
Ach1 0.77 0.60 0.29 0.24 0.20
Ach2 0.60 0.46 0.22 0.18 0.16
Amb1 0.29 0.22 0.85 0.70 0.60
Amb2 0.24 0.18 0.70 0.58 0.49
Amb3 0.20 0.16 0.60 0.49 0.42
```

Model = P r P’

Residual = Data - Model
Data = Model + Error

round(resid,2)

	Ach1	Ach2	Amb1	Amb2	Amb3
Ach1	0.23	0.00	0.01	-0.04	0.00
Ach2	0.00	0.54	-0.02	0.12	-0.06
Amb1	0.01	-0.02	0.15	0.00	0.00
Amb2	-0.04	0.12	0.00	0.42	0.01
Amb3	0.00	-0.06	0.00	0.01	0.58

Estimating the goodness of fit

I. How big are the residuals?

   A. compared to what?

      1. Deviation from a 0 matrix of sample size N
      2. as a function of the number of parameters
      3. as a function of phases of the moon

Alternative goodness of fit indices

```
Model Chisquare =  9.74   Df =  4 Pr(>Chisq) = 0.0450
Chisquare (null model) =  176   Df =  10
Goodness-of-fit index =  0.964
Adjusted goodness-of-fit index =  0.865
RMSEA index =  0.120   90% CI: (0.0164, 0.219)
Bentler-Bonnett NFI =  0.945
Tucker-Lewis NNFI =  0.914
Bentler CFI =  0.965
BIC =  -8.68

Normalized Residuals
    Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
-5.74e-01 -3.76e-02 -2.03e-06  4.83e-03  3.85e-05  1.13e+00
```