Psychology 405: Psychometric Theory

William Revelle
Northwestern University

Spring, 2013

http://personality-project.org/revelle/syllabi/405.syllabus.html
and
http://personality-project.org/revelle/syllabi/405.old.syllabus.html
Lord Kelvin’s dictum

In physical science a first essential step in the direction of learning any subject is to find principles of numerical reckoning and methods for practicably measuring some quality connected with it. I often say that when you can measure what you are speaking about and express it in numbers you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the stage of science, whatever the matter may be. (Thomsom, 1891)

Taken from Michell (2003) in his critique of psychometrics:
Psychometric Theory

- ‘The character which shapes our conduct is a definite and durable ‘something’, and therefore ... it is reasonable to attempt to measure it. (Galton, 1884)
- “Whatever exists at all exists in some amount. To know it thoroughly involves knowing its quantity as well as its quality” (E.L. Thorndike, 1918)
Psychology and the need for measurement

• “The history of science is the history of measurement” (J. M. Cattell, 1893)

• “We hardly recognize a subject as scientific if measurement is not one of its tools” (Boring, 1929)

• “There is yet another [method] so vital that, if lacking it, any study is thought ... not be scientific in the full sense of the word. This further an crucial method is that of measurement.” (Spearman, 1937)

• “One’s knowledge of science begins when he can measure what he is speaking about and express in numbers” (Eysenck, 1973)
Psychometric Theory: Goals

1. To acquire the fundamental vocabulary and logic of psychometric theory.
2. To develop your capacity for critical judgment of the adequacy of measures purported to assess psychological constructs.
3. To acquaint you with some of the relevant literature in personality assessment, psychometric theory and practice, and methods of observing and measuring affect, behavior, cognition and motivation.
Psychometric Theory: Goals II

1. To instill an appreciation of and an interest in the principles and methods of psychometric theory.

2. This course is not designed to make you into an accomplished psychometrist (one who gives tests) nor is it designed to make you a skilled psychometrician (one who constructs tests).

3. It will give you limited experience with psychometric computer programs (although all of the examples will use R, it not necessary to learn R).
Psychometric Theory: Requirements

- Asking questions!
- Objective Midterm exam
- Objective Final exam
- Final paper applying principles from the course to a problem of interest to you.
- Sporadic applied homework and data sets
Text and Syllabus

- Revelle, W. *An introduction to psychometric theory with applications in R* (under development - see web)
  - [http://personality-project.org/r/book](http://personality-project.org/r/book)
- web guide to class:
  - [http://personality-project.org/revelle/syllabi/405.syllabus.html](http://personality-project.org/revelle/syllabi/405.syllabus.html)
Syllabus: Overview

I. Individual Differences and Experimental Psychology
II. Models of measurement
III. Test theory
   A. Reliability
   B. Validity (predictive and construct)
   C. Structural Models
   D. Test Construction
IV. Assessment of traits
V. Methods of observation of behavior
Psychometric Theory: A conceptual Syllabus

- X1
- X2
- X3
- X4
- X5
- X6
- X7
- X8
- X9

- L1
- L2
- L3
- L4
- L5

- Y1
- Y2
- Y3
- Y4
- Y5
- Y6
- Y7
- Y8
Constructs/Latent Variables

L1
L2
L3
L4
L5
Examples of psychological constructs

• Anxiety
  – Trait
  – State
• Love
• Conformity
• Intelligence
• Learning and memory
  – Procedural - memory for how
  – Episodic -- memory for what
  • Implicit
  • explicit
• ...
Theory as organization of constructs

L1 → L4
L2 → L4
L3 → L5 → L4
L5 → L4
Theories as metaphors and analogies-1

• Physics
  – Planetary motion
    • Ptolemy
    • Galileo
    • Einstein
  – Springs, pendulums, and electrical circuits
  – The Bohr atom

• Biology
  – Evolutionary theory
  – Genetic transmission
Theories as metaphors and analogies-2

• Business competition and evolutionary theory
  – Business niche
  – Adaptation to change in niches
• Learning, memory, and cognitive psychology
  – Telephone as an example of wiring of connections
  – Digital computer as information processor
  – Parallel processes as distributed information processor
Models and theory

• Formal models
  – Mathematical models
  – Dynamic models - simulations
• Conceptual models
  – As guides to new research
  – As ways of telling a story
  • Organizational devices
Observable or measured variables

X1  Y1
X2  Y2
X3  Y3
X4  Y4
X5  Y5
X6  Y6
X7  Y7
X8  Y8
X9
Observed Variables

- Item Endorsement
- Reaction time
- Choice/Preference
- Blood Oxygen Level Dependent Response
- Skin Conductance
- Archival measures
Theory development and testing

- Theories as organizations of observable variables
- Constructs, latent variables and observed variables
  - Observable variables
    - Multiple levels of description and abstraction
    - Multiple levels of inference about observed variables
  - Latent Variables
    - Latent variables as the common theme of a set of observables
    - Central tendency across time, space, people, situations
  - Constructs as organizations of latent variables and observed variables
Psychometric Theory: A conceptual Syllabus
A Theory of Data: What can be measured

What is measured?
- Objects
- Individuals

What kind of measures are taken?
- Order
- Proximity

What kind of comparisons are made?
- Single Dyads
- Pairs of Dyads
Scaling:
the mapping between observed and latent variables
Variance, Covariance, and Correlation

Simple correlation

Simple regression

Multiple correlation/regression

Partial correlation
Techniques of Data Reduction: Factor and Components Analysis

X1
X2
X3
X4
X5
X6
X7
X8
X9

L1

L2

L3

L4

L5

Y1
Y2
Y3
Y4
Y5
Y6
Y7
Y8
Classic Reliability Theory: How well do we measure what ever we are measuring

\[ X_1 \rightarrow L_1 \rightarrow X_2 \rightarrow X_3 \]
Modern Reliability Theory: Item Response Theory

How well do we measure whatever we are measuring?

Performance as a function of Ability and Test Difficulty

X1
X2
X3

Ability
Performance

Easy
Difficult
Types of Validity: What are we measuring

- **Face**
  - X1
  - Concurrent
    - X2
    - Predictive
      - X3
      - Construct
        - X4
        - Convergent
          - X5
          - Discriminant
            - X6
            - L2
            - L3
            - L5
  - Y1
  - Y2
  - Y3

- **Convergent**
  - L2
  - Y4

- **Discriminant**
  - L3
  - Y5

- **L5**
  - Y6
  - Y7
  - Y8
Structural Equation Modeling: Combining Measurement and Structural Models
Scale Construction: practical and theoretical
Traits and States: What is measured?

- X1
- X2
- X3
- X4
- X5
- X6
- X7
- X8
- X9

- BAS
- BIS
- IQ
- Affect
- Cog

- Y1
- Y2
- Y3
- Y4
- Y5
- Y6
- Y7
- Y8
The data box: measurement across time, situations, items, and people
Psychometric Theory: A conceptual Syllabus
Syllabus: Overview

I. Individual Differences and Experimental Psychology
   A. Two historic approaches to the study of psychology
   B. Individual differences and general laws
   C. The two disciplines reconsidered

II. Models of measurement
   A. Theory of Data
   B. Issues in scaling
   C. Variance, Covariance, and Correlation
   D. Dimension reduction: Factor, Component and Cluster analysis

III. Test theory
   A. Reliability
   B. Validity (predictive and construct)
   C. Structural Models
   D. Test Construction

IV. Assessment of traits

V. Methods of observation of behavior
# Two Disciplines of Psychological Research


<table>
<thead>
<tr>
<th>$B = f(\text{Personality})$</th>
<th>$B = f(\text{P*E})$</th>
<th>$B = f(\text{Environment})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Darwin</td>
<td></td>
</tr>
<tr>
<td>Galton</td>
<td>Fechner, Weber, Wundt</td>
<td></td>
</tr>
<tr>
<td>Binet, Terman</td>
<td>Watson, Thorndike</td>
<td></td>
</tr>
<tr>
<td>Allport, Burt</td>
<td>Lewin</td>
<td>Hull, Tolman</td>
</tr>
<tr>
<td>Cattell</td>
<td>Atkinson, Eysenck</td>
<td>Spence, Skinner</td>
</tr>
<tr>
<td>Epstein, Norman, Goldberg, Costa, McCrae</td>
<td></td>
<td>Mischel Cervone</td>
</tr>
</tbody>
</table>
## Two Disciplines of Psychological Research

<table>
<thead>
<tr>
<th>Method/Model</th>
<th>B=f(Person)</th>
<th>B=f(Environment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method/Model</td>
<td>Correlational</td>
<td>Experimental</td>
</tr>
<tr>
<td></td>
<td>Observational</td>
<td>Causal</td>
</tr>
<tr>
<td></td>
<td>Biological/field</td>
<td>Physical/lab</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Variance</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dispersion</td>
<td>Central Tendency</td>
</tr>
<tr>
<td></td>
<td>Correlation/ Covariance</td>
<td>t-test, F test</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Effects</th>
<th>Individuals</th>
<th>Situations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Individual Differences</td>
<td>General Laws</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B=f(P,E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of individual in an environment</td>
</tr>
</tbody>
</table>

Multivariate Experimental Psychology
Experimental Personality Research involves theory, measurement and experimental technique.
Experimental Personality Research involves theory, measurement and experimental technique.
Experimental Personality Research involves theory, measurement and experimental technique.
Theory and Theory Testing I: Theory

Construct 1 → Construct 2
Theory and Theory Testing II: Experimental manipulation

Construct 1 \(\rightarrow\) Construct 2

\[ r_{c1c2} \]

Manipulation 1 \(\rightarrow\) Observation 1

\[ r_{mo} \]

\[ F_m \]
Correlational inference

Construct 1

Observation 1

Construct 2

Observation 2

$r_{oo}$
Theory and Theory Testing IV: Correlational inference

- Construct 1
- Observation 1
- Construct X
- Construct 2
- Observation 2

? r_{oo}
Theory and Theory Testing V: Alternative Explanations

Construct 1

Observation 1

Construct 2

Observation 2
Individual differences and general laws

- Impulsivity
- Arousal
- Attention
- Working Memory
- Reaction Time
- GREs
- Memory Span
- Caffeine
Theory and Theory Testing VI: Eliminate Alternative Explanations

Construct 1

Construct 2

Observation 1

Observation 2
Types of Relationships

(Vale and Vale, 1969)

- Behavior = f(Situation)
- Behavior = f₁(Situation) + f₂(Personality)
- Behavior = f₁(Situation) + f₂(Personality) + f₃(Situation*Personality)
- Behavior = f₁(Situation * Personality)
- Behavior = idiosyncratic
Types of Relationships: 
Behavior = f(Situation)

Neuronal excitation = f(light intensity)
Types of Relationships: 
Behavior = \( f_1(\text{Situation}) + f_2(\text{Person}) \)

Environmental Input (income)
Probability of college = \( f_1(\text{income}) + f_2(\text{ability}) \)
Types of Relationships:

Behavior = f1(Situation) + f2(Personality) + f3(Situation*Personality)

Avoidance = f1(shock intensity) + f2(anxiety) + f3(shock*anxiety)
Reading = f1(sesame street) = f2(ability) + f3(ss * ability)
Types of Relationships:
Behavior = f(Situation*Person)

Eating = f(preload * restraint)
GRE = f(caffeine * impulsivity)
Types of Relationships:
Behavior = f(Situation*Person)

Environmental Input
GRE = f(caffeine * impulsivity)
Persons, Situations, and Theory

Theoretical model

External stimulation $\rightarrow$ Arousal $\rightarrow$ Performance

Individual Difference

General Law

Arousal $\rightarrow$ Performance

Observed relationship
Psychometric Theory: A conceptual Syllabus
Data = Model + Residual

In all of psychometrics and statistics, five questions to ask are:

1. What is the model?
2. How well does it fit?
3. What are the plausible alternative models?
4. How well do they fit?
5. Is this better or worse than the current fit?
A Theory of Data: What can be measured

What is measured?
Objects
Individuals

What kind of measures are taken?
Proximity (- distance)
Order

What kind of comparisons are made?
Single Dyads
Pairs of Dyads
Assigning numbers to observations

<table>
<thead>
<tr>
<th>2.718281828459050</th>
<th>3,412.1416</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.141592653589790</td>
<td>86,400</td>
</tr>
<tr>
<td>24</td>
<td>31,557,600</td>
</tr>
<tr>
<td>37</td>
<td>299,792,458</td>
</tr>
<tr>
<td>98.7</td>
<td>6.022141 \times 10^{23}</td>
</tr>
<tr>
<td>365.25</td>
<td>42</td>
</tr>
<tr>
<td>365.25636305</td>
<td>X</td>
</tr>
</tbody>
</table>
Assigning numbers to observations: order vs. proximity

- Suppose we have observations X, Y, Z
- We assume each observation is a point on an attribute dimension (see Michell for a critique of the assumption of quantity).
- Assign a number to each point.
- Two questions to ask:
  - What is the order of the points?
  - How far apart are the points?
Scaling of objects

• Consider $O = \{ o_1, o_2, \ldots, o_n \}$ and

• $O \times O = \{(o_1, o_1), (o_1, o_2), \ldots (o_1, o_n), \ldots, (o_2, o_n), \ldots,$

$(o_n, o_n)\}$

• Can we assign scale values to objects that satisfy an order relationship “≤”

• $o_i \leq o_j$ and $o_i \geq o_j \iff o_i = o_j$

• $o_i \leq o_j$ and $o_j \leq o_k \iff o_i \leq o_k$ (transitive)
Moh’s index of hardness

<table>
<thead>
<tr>
<th>Mohs Hardness</th>
<th>Mineral</th>
<th>Scratch hardness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Talc</td>
<td>.59</td>
</tr>
<tr>
<td>2</td>
<td>Gypsum</td>
<td>.61</td>
</tr>
<tr>
<td>3</td>
<td>Calcite</td>
<td>3.44</td>
</tr>
<tr>
<td>4</td>
<td>Fluorite</td>
<td>3.05</td>
</tr>
<tr>
<td>5</td>
<td>Apapatite</td>
<td>5.2</td>
</tr>
<tr>
<td>6</td>
<td>Orthoclase Feldspar</td>
<td>37.2</td>
</tr>
<tr>
<td>7</td>
<td>Quartz</td>
<td>100</td>
</tr>
<tr>
<td>8</td>
<td>Topaz</td>
<td>121</td>
</tr>
<tr>
<td>9</td>
<td>Corundum</td>
<td>949</td>
</tr>
<tr>
<td>10</td>
<td>Diamond</td>
<td>85,300</td>
</tr>
</tbody>
</table>

Note the strong non-linearity of the top end of the scale.
The Beaufort Scale

<table>
<thead>
<tr>
<th>Force</th>
<th>Wind (Knots)</th>
<th>WMO Classification</th>
<th>Appearance of Wind Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Less than 1</td>
<td>Calm</td>
<td>Sea surface smooth and mirror-like</td>
</tr>
<tr>
<td>1</td>
<td>1-3</td>
<td>Light Air</td>
<td>Scaly ripples, no foam crests</td>
</tr>
<tr>
<td>2</td>
<td>4-6</td>
<td>Light Breeze</td>
<td>Small wavelets, crests glassy, no breaking</td>
</tr>
<tr>
<td>3</td>
<td>7-10</td>
<td>Gentle Breeze</td>
<td>Large wavelets, crests begin to break, scattered whitecaps</td>
</tr>
<tr>
<td>4</td>
<td>11-16</td>
<td>Moderate Breeze</td>
<td>Small waves 1-4 ft. becoming longer, numerous whitecaps</td>
</tr>
<tr>
<td>5</td>
<td>17-21</td>
<td>Fresh Breeze</td>
<td>Moderate waves 4-8 ft taking longer form, many whitecaps, some spray</td>
</tr>
<tr>
<td>6</td>
<td>22-27</td>
<td>Strong Breeze</td>
<td>Larger waves 8-13 ft, whitecaps common more spray</td>
</tr>
<tr>
<td>7</td>
<td>28-33</td>
<td>Near Gale</td>
<td>Sea heaps up, waves 13-20 ft, white foam streaks off breakers</td>
</tr>
<tr>
<td>8</td>
<td>34-40</td>
<td>Gale Moderately</td>
<td>High (13-20 ft) waves of greater length, edges of crests begin to break into spindrift, foam blown in streaks</td>
</tr>
<tr>
<td>9</td>
<td>41-47</td>
<td>Strong Gale</td>
<td>High waves (20 ft), sea begins to roll, dense streaks of foam, spray may reduce visibility</td>
</tr>
<tr>
<td>10</td>
<td>48-55</td>
<td>Storm</td>
<td>Very high waves (20-30 ft) with overhanging crests, sea white with densely blown foam, heavy rolling, lowered visibility</td>
</tr>
<tr>
<td>11</td>
<td>56-63</td>
<td>Violent Storm</td>
<td>Exceptionally high (30-45 ft) waves, foam patches cover sea, visibility more reduced</td>
</tr>
<tr>
<td>12</td>
<td>64+</td>
<td>Hurricane</td>
<td>Air filled with foam, waves over 45 ft, sea completely white with driving spray, visibility greatly reduced</td>
</tr>
</tbody>
</table>

Roughly linear with windspeed, but force of wind is quadratic effect of wind speed
Scaling of objects subjects as replicates

- Typical object scaling is concerned with order or location of objects
- Subjects are assumed to be random replicates of each other, differing only as a source of noise
Absolute scaling techniques

- “On a scale from 1 to 10” this ... is a ___?
- If A is 1 and B is 10, then what is C?
- College rankings based upon selectivity
- College rankings based upon “yield”
- Zagat ratings of restaurants
Absolute scaling difficulties

- “On a scale from 1 to 10” this ... is a ___?
- sensitive to context effects
  - what if a new object appears?
- Need unbounded scale
- If A is 1 and B is 10, then what is C?
  - results will depend upon A, B
Absolute scaling: artifacts

• College rankings based upon selectivity
  • accept/applied
    • encourage less able to apply
• College rankings based upon “yield”
  • matriculate/accepted
    • early admissions guarantee matriculation
  • don’t accept students who will not attend
College admission tricks
Increase the yield by rejecting students likely to go elsewhere.

(see also Avery, C., Glickman, M, Hoxby, C., & Metrick, A, 2013)
Models of scaling objects

• Assume each object (a, b,...z) has a scale value (A, B, ... Z) with some noise for each measurement.

• Probability of $A > B$ increases with difference of between $a$ and $b$

  • $P(A>B) = f(a - b)$

• Can we find a function, $f$, such that equal differences in the latent variable (a, b, c) lead to equal differences in the observed variable?
Models of scaling

- Given latent scores \((a, b, \ldots, z)\) find observed scores \(A = f(a), B = f(b), \ldots, Z = f(z)\) such that iff \(a > b\) then \(A > B\) (an ordinal scale)

- Given latent scores \((a, b, \ldots, z)\) find observed scores \(A = f(a), B = f(b), \ldots, Z = f(z)\) such that iff \(a - b > c - d\) then \(A - B > C - D\) (an interval scale)

- Given latent scores \((a, b, \ldots, z)\) find observed scores \(A = f(a), B = f(b), \ldots, Z = f(z)\) such that iff \(a/b > c/d\) then \(A/B > C/D\) (a ratio scale)
Thurstonian Scaling of Stimuli

- What is scale location of objects I and J on an attribute dimension D?
- Assume that object I has mean value $m_i$ with some variability.
- Assume that object J has a mean value $m_j$
- Assume equal and normal variability (Thurstone case 5)
  - Less restrictive assumptions are cases 1-4
- Observe frequency of ($o_i < o_j$)
- Convert relative frequencies to normal equivalents
- Result is an interval scale with arbitrary 0 point
Thurstonian Scaling

Frequency

<table>
<thead>
<tr>
<th></th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>0.5</td>
<td>0.7</td>
<td>0.9</td>
</tr>
<tr>
<td>x2</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>x3</td>
<td>0.1</td>
<td>0.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$p(col > row)$
Thurstone comparative judgment

![Graphs showing response strength and probability of choice against latent scale value.](image)
Thurstone scaling: step 1: choice

```r
> data(vegetables)
> round(veg, 2)
```

<table>
<thead>
<tr>
<th></th>
<th>Turn</th>
<th>Cab</th>
<th>Beet</th>
<th>Asp</th>
<th>Car</th>
<th>Spin</th>
<th>S.Beans</th>
<th>Peas</th>
<th>Corn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>0.50</td>
<td>0.82</td>
<td>0.77</td>
<td>0.81</td>
<td>0.88</td>
<td>0.89</td>
<td>0.90</td>
<td>0.89</td>
<td>0.93</td>
</tr>
<tr>
<td>Cab</td>
<td>0.18</td>
<td>0.50</td>
<td>0.60</td>
<td>0.72</td>
<td>0.74</td>
<td>0.74</td>
<td>0.81</td>
<td>0.84</td>
<td>0.86</td>
</tr>
<tr>
<td>Beet</td>
<td>0.23</td>
<td>0.40</td>
<td>0.50</td>
<td>0.56</td>
<td>0.74</td>
<td>0.68</td>
<td>0.84</td>
<td>0.80</td>
<td>0.82</td>
</tr>
<tr>
<td>Asp</td>
<td>0.19</td>
<td>0.28</td>
<td>0.44</td>
<td>0.50</td>
<td>0.56</td>
<td>0.59</td>
<td>0.68</td>
<td>0.60</td>
<td>0.73</td>
</tr>
<tr>
<td>Car</td>
<td>0.12</td>
<td>0.26</td>
<td>0.26</td>
<td>0.44</td>
<td>0.50</td>
<td>0.49</td>
<td>0.57</td>
<td>0.71</td>
<td>0.76</td>
</tr>
<tr>
<td>Spin</td>
<td>0.11</td>
<td>0.26</td>
<td>0.32</td>
<td>0.41</td>
<td>0.51</td>
<td>0.50</td>
<td>0.63</td>
<td>0.68</td>
<td>0.63</td>
</tr>
<tr>
<td>S.Beans</td>
<td>0.10</td>
<td>0.19</td>
<td>0.16</td>
<td>0.32</td>
<td>0.43</td>
<td>0.37</td>
<td>0.50</td>
<td>0.53</td>
<td>0.64</td>
</tr>
<tr>
<td>Peas</td>
<td>0.11</td>
<td>0.16</td>
<td>0.20</td>
<td>0.40</td>
<td>0.29</td>
<td>0.32</td>
<td>0.47</td>
<td>0.50</td>
<td>0.63</td>
</tr>
<tr>
<td>Corn</td>
<td>0.07</td>
<td>0.14</td>
<td>0.18</td>
<td>0.27</td>
<td>0.24</td>
<td>0.37</td>
<td>0.36</td>
<td>0.37</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Thurstone scaling: step 2: normal transformation

\[
\text{normal.values} <- \text{qnorm(as.matrix(veg))}
\]
\[
\text{round(normal.values,2)}
\]

<table>
<thead>
<tr>
<th></th>
<th>Turn</th>
<th>Cab</th>
<th>Beet</th>
<th>Asp</th>
<th>Car</th>
<th>Spin</th>
<th>S.Beans</th>
<th>Peas</th>
<th>Corn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>0.00</td>
<td>0.91</td>
<td>0.74</td>
<td>0.88</td>
<td>1.17</td>
<td>1.24</td>
<td>1.28</td>
<td>1.24</td>
<td>1.45</td>
</tr>
<tr>
<td>Cab</td>
<td>-0.91</td>
<td>0.00</td>
<td>0.26</td>
<td>0.59</td>
<td>0.65</td>
<td>0.63</td>
<td>0.88</td>
<td>1.02</td>
<td>1.07</td>
</tr>
<tr>
<td>Beet</td>
<td>-0.74</td>
<td>-0.26</td>
<td>0.00</td>
<td>0.15</td>
<td>0.63</td>
<td>0.46</td>
<td>1.02</td>
<td>0.83</td>
<td>0.91</td>
</tr>
<tr>
<td>Asp</td>
<td>-0.88</td>
<td>-0.59</td>
<td>-0.15</td>
<td>0.00</td>
<td>0.15</td>
<td>0.22</td>
<td>0.46</td>
<td>0.26</td>
<td>0.61</td>
</tr>
<tr>
<td>Car</td>
<td>-1.17</td>
<td>-0.65</td>
<td>-0.63</td>
<td>-0.15</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.19</td>
<td>0.55</td>
<td>0.72</td>
</tr>
<tr>
<td>Spin</td>
<td>-1.24</td>
<td>-0.63</td>
<td>-0.46</td>
<td>-0.22</td>
<td>0.02</td>
<td>0.00</td>
<td>0.33</td>
<td>0.47</td>
<td>0.33</td>
</tr>
<tr>
<td>SBeans</td>
<td>-1.28</td>
<td>-0.88</td>
<td>-1.02</td>
<td>-0.46</td>
<td>-0.19</td>
<td>-0.33</td>
<td>0.00</td>
<td>0.07</td>
<td>0.36</td>
</tr>
<tr>
<td>Peas</td>
<td>-1.24</td>
<td>-1.02</td>
<td>-0.83</td>
<td>-0.26</td>
<td>-0.55</td>
<td>-0.47</td>
<td>-0.07</td>
<td>0.00</td>
<td>0.33</td>
</tr>
<tr>
<td>Corn</td>
<td>-1.45</td>
<td>-1.07</td>
<td>-0.91</td>
<td>-0.61</td>
<td>-0.72</td>
<td>-0.33</td>
<td>-0.36</td>
<td>-0.33</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Thurstone Step 3: average z score and rescale

Sums of z scores
Average z score
Rescale to set minimum to 0

<table>
<thead>
<tr>
<th></th>
<th>sums</th>
<th>means</th>
<th>values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>-8.89</td>
<td>-0.99</td>
<td>0.00</td>
</tr>
<tr>
<td>Cab</td>
<td>-4.19</td>
<td>-0.47</td>
<td>0.52</td>
</tr>
<tr>
<td>Beet</td>
<td>-3.00</td>
<td>-0.33</td>
<td>0.65</td>
</tr>
<tr>
<td>Asp</td>
<td>-0.07</td>
<td>-0.01</td>
<td>0.98</td>
</tr>
<tr>
<td>Car</td>
<td>1.16</td>
<td>0.13</td>
<td>1.12</td>
</tr>
<tr>
<td>Spin</td>
<td>1.40</td>
<td>0.16</td>
<td>1.14</td>
</tr>
<tr>
<td>S.Beans</td>
<td>3.71</td>
<td>0.41</td>
<td>1.40</td>
</tr>
<tr>
<td>Peas</td>
<td>4.10</td>
<td>0.46</td>
<td>1.44</td>
</tr>
<tr>
<td>Corn</td>
<td>5.77</td>
<td>0.64</td>
<td>1.63</td>
</tr>
</tbody>
</table>

> sums <- colSums(normal.values)
> means <- colMeans(normal.values)
> values <- means - min(values)
> thurstone.df <- data.frame(sums,means,values)
> round(thurstone.df,2)

Goodness of fit test: 1-residual^2/original^2 = .99
Create a function to do this

> thurstone
function (x, ranks = FALSE, digits = 2)
{
  cl <- match.call()
  if (ranks) {
    choice <- choice.mat(x)
  }
  else {
    if (is.matrix(x))
      choice <- x
    choice <- as.matrix(x)
  }
  scale.values <- colMeans(qnorm(choice)) -
  min(colMeans(qnorm(choice)))
  model <- pnorm(-scale.values % +% t(scale.values))
  error <- model - choice
  fit <- 1 - (sum(error * error)/sum(choice * choice))
  result <- list(scale = round(scale.values, digits), GF = fit,
    residual = error, Call = cl)
  class(result) <- c("psych", "thurstone")
  return(result)
}
Thurstone Model

```r
> veg.scale <- thurstone(veg)  #Apply our new function
> veg.scale
Thurstonian scale (case 5) scale values
Call: thurstone(x = veg)
  Turn   Cab   Beet    Asp    Car   Spin S.Beans   Peas   Corn
  0.00  0.52  0.65   0.98   1.12   1.14   1.40   1.44   1.63

  Goodness of fit of model  0.99

> values <- veg.scale$scale
> model <- -values %+% t(values)
> colnames(model) <- rownames(model) <- names(values)
> model

       Turn   Cab  Beet   Asp  Car  Spin S.Beans  Peas  Corn
  Turn    0.00  0.52  0.65  0.98  1.12  1.14   1.40  1.44  1.63
  Cab   -0.52  0.00  0.13  0.46  0.60  0.62   0.88  0.92  1.11
  Beet   -0.65 -0.13  0.00  0.33  0.47  0.49   0.75  0.79  0.98
  Asp   -0.98 -0.46 -0.33  0.00  0.14  0.16   0.42  0.46  0.65
  Car   -1.12 -0.60 -0.47 -0.14  0.00  0.02   0.28  0.32  0.51
  Spin  -1.14 -0.62 -0.49 -0.16 -0.02  0.00   0.26  0.30  0.49
S.Beans -1.40 -0.88 -0.75 -0.42 -0.28 -0.26  0.00  0.04  0.23
  Peas  -1.44 -0.92 -0.79 -0.46 -0.32 -0.30  -0.04  0.00  0.19
  Corn  -1.63 -1.11 -0.98 -0.65 -0.51 -0.49  -0.23 -0.19  0.00
```
Thurstone: model

> model <- pnorm(model) # convert Z scores to probability
> round(model,2)

<table>
<thead>
<tr>
<th></th>
<th>Turn</th>
<th>Cab</th>
<th>Beet</th>
<th>Asp</th>
<th>Car</th>
<th>Spin</th>
<th>S.Beans</th>
<th>Peas</th>
<th>Corn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>0.50</td>
<td>0.70</td>
<td>0.74</td>
<td>0.84</td>
<td>0.87</td>
<td>0.87</td>
<td>0.92</td>
<td>0.93</td>
<td>0.95</td>
</tr>
<tr>
<td>Cab</td>
<td>0.30</td>
<td>0.50</td>
<td>0.55</td>
<td>0.68</td>
<td>0.73</td>
<td>0.73</td>
<td>0.81</td>
<td>0.82</td>
<td>0.87</td>
</tr>
<tr>
<td>Beet</td>
<td>0.26</td>
<td>0.45</td>
<td>0.50</td>
<td>0.63</td>
<td>0.68</td>
<td>0.69</td>
<td>0.77</td>
<td>0.79</td>
<td>0.84</td>
</tr>
<tr>
<td>Asp</td>
<td>0.16</td>
<td>0.32</td>
<td>0.37</td>
<td>0.50</td>
<td>0.56</td>
<td>0.56</td>
<td>0.66</td>
<td>0.68</td>
<td>0.74</td>
</tr>
<tr>
<td>Car</td>
<td>0.13</td>
<td>0.27</td>
<td>0.32</td>
<td>0.44</td>
<td>0.50</td>
<td>0.51</td>
<td>0.61</td>
<td>0.63</td>
<td>0.69</td>
</tr>
<tr>
<td>Spin</td>
<td>0.13</td>
<td>0.27</td>
<td>0.31</td>
<td>0.44</td>
<td>0.49</td>
<td>0.50</td>
<td>0.60</td>
<td>0.62</td>
<td>0.69</td>
</tr>
<tr>
<td>S.Beans</td>
<td>0.08</td>
<td>0.19</td>
<td>0.23</td>
<td>0.34</td>
<td>0.39</td>
<td>0.40</td>
<td>0.50</td>
<td>0.52</td>
<td>0.59</td>
</tr>
<tr>
<td>Peas</td>
<td>0.07</td>
<td>0.18</td>
<td>0.21</td>
<td>0.32</td>
<td>0.37</td>
<td>0.38</td>
<td>0.48</td>
<td>0.50</td>
<td>0.58</td>
</tr>
<tr>
<td>Corn</td>
<td>0.05</td>
<td>0.13</td>
<td>0.16</td>
<td>0.26</td>
<td>0.31</td>
<td>0.31</td>
<td>0.41</td>
<td>0.42</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Thurstone: Data-Model

```r
> error <- veg - model
> round(error,2)
```

<table>
<thead>
<tr>
<th></th>
<th>Turn</th>
<th>Cab</th>
<th>Beet</th>
<th>Asp</th>
<th>Car</th>
<th>Spin</th>
<th>S.Beans</th>
<th>Peas</th>
<th>Corn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>0.00</td>
<td>0.12</td>
<td>0.03</td>
<td>-0.03</td>
<td>0.01</td>
<td>0.02</td>
<td>-0.02</td>
<td>-0.03</td>
<td>-0.02</td>
</tr>
<tr>
<td>Cab</td>
<td>-0.12</td>
<td>0.00</td>
<td>0.05</td>
<td>0.05</td>
<td>0.02</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>-0.01</td>
</tr>
<tr>
<td>Beet</td>
<td>-0.03</td>
<td>-0.05</td>
<td>0.00</td>
<td>-0.07</td>
<td>0.06</td>
<td>-0.01</td>
<td>0.07</td>
<td>0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>Asp</td>
<td>0.03</td>
<td>-0.05</td>
<td>0.07</td>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
<td>-0.08</td>
<td>-0.01</td>
</tr>
<tr>
<td>Car</td>
<td>-0.01</td>
<td>-0.02</td>
<td>-0.06</td>
<td>-0.01</td>
<td>0.00</td>
<td>-0.02</td>
<td>-0.04</td>
<td>0.08</td>
<td>0.07</td>
</tr>
<tr>
<td>Spin</td>
<td>-0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.03</td>
<td>0.06</td>
<td>-0.06</td>
</tr>
<tr>
<td>S.Beans</td>
<td>0.02</td>
<td>0.00</td>
<td>-0.07</td>
<td>-0.01</td>
<td>0.04</td>
<td>-0.03</td>
<td>0.00</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>Peas</td>
<td>0.03</td>
<td>-0.02</td>
<td>-0.01</td>
<td>0.08</td>
<td>-0.08</td>
<td>-0.06</td>
<td>-0.01</td>
<td>0.00</td>
<td>0.05</td>
</tr>
<tr>
<td>Corn</td>
<td>0.02</td>
<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
<td>-0.07</td>
<td>0.06</td>
<td>-0.05</td>
<td>-0.05</td>
<td>0.00</td>
</tr>
</tbody>
</table>
### Summarize Residuals

```r
describe(error, skew=FALSE)
```

<table>
<thead>
<tr>
<th>var</th>
<th>n</th>
<th>mean</th>
<th>sd</th>
<th>median</th>
<th>trimmed</th>
<th>mad</th>
<th>min</th>
<th>max</th>
<th>range</th>
<th>se</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>1</td>
<td>0.9</td>
<td>0.05</td>
<td>0.00</td>
<td>-0.01</td>
<td>0.03</td>
<td>-0.12</td>
<td>0.03</td>
<td>0.15</td>
<td>0.02</td>
</tr>
<tr>
<td>Cab</td>
<td>2</td>
<td>0.0</td>
<td>0.05</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>-0.05</td>
<td>0.12</td>
<td>0.17</td>
<td>0.02</td>
</tr>
<tr>
<td>Beet</td>
<td>3</td>
<td>0.0</td>
<td>0.05</td>
<td>0.01</td>
<td>0.00</td>
<td>0.04</td>
<td>-0.07</td>
<td>0.07</td>
<td>0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>Asp</td>
<td>4</td>
<td>0.0</td>
<td>0.04</td>
<td>-0.01</td>
<td>0.00</td>
<td>0.03</td>
<td>-0.07</td>
<td>0.08</td>
<td>0.14</td>
<td>0.01</td>
</tr>
<tr>
<td>Car</td>
<td>5</td>
<td>0.0</td>
<td>0.05</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>-0.08</td>
<td>0.06</td>
<td>0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>Spin</td>
<td>6</td>
<td>0.0</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>-0.06</td>
<td>0.06</td>
<td>0.12</td>
<td>0.01</td>
</tr>
<tr>
<td>S.Beans</td>
<td>7</td>
<td>0.0</td>
<td>0.04</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>-0.05</td>
<td>0.07</td>
<td>0.12</td>
<td>0.01</td>
</tr>
<tr>
<td>Peas</td>
<td>8</td>
<td>0.0</td>
<td>0.05</td>
<td>0.01</td>
<td>0.00</td>
<td>0.06</td>
<td>-0.08</td>
<td>0.08</td>
<td>0.16</td>
<td>0.02</td>
</tr>
<tr>
<td>Corn</td>
<td>9</td>
<td>0.01</td>
<td>0.04</td>
<td>-0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>-0.06</td>
<td>0.07</td>
<td>0.13</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Find fit

- Many indices of fit
- Typical is $1 - \frac{\text{error}^2}{\text{data}^2}$
- Fit of Thurstone =
  - $> 1 - \frac{\text{sum}(\text{error}^2)}{\text{sum}(\text{veg}^2)}$
  - [1] 0.99
Alternative scaling models

- Thurstone assumes normal deviations
- Logistic model produces similar results
  - used in scaling chess players, sports teams
    - win/loss record
  - scaling of colleges by where students choose to go (choice of A vs. B)
    - more difficult to fake
Compare to other scaling methods

- Thurstone assumes normal error of preference
- logistic model is alternative model
- so are other rank difference models
- all about the same in terms of fit
At least two ways to collect choice data

- Paired comparisons:
  - Is X > Y
  - Is Y > Z
  - ... n*(n-1)/2 pairs

- Rank orders (X>Y>Z>W) => a set of pairs
  - X>Y, X>Z, Y>Z, X>W, Y> W, Z>W
Thurstonian scaling in R

- code for Thurstonian case V is in psych
  - data(vegetables)
  - thu <- thurstone(veg)
  - thu  #shows values and fits
  - thu$residual  #shows residuals
- brief discussion of Thurstonian and alternative scaling models with links at
  - http://personality-project.org/r/thurstone.html
What is this thing called R?

- A quick introduction to R: gettingstarted
- personality-project.org/r/psych
Assigning numbers: do they form a metric space?

- Suppose we have observations X, Y, Z
- We assume each observation is a point on (possibly many) attribute dimension(s)
- Assign a number to each point.
- Do these numbers form a metric space?
- Requires finding a distance between points
Metric spaces and the axioms of a distance measure

- A metric space is a set of points with a distance function, D, which meets the following properties:
  - Distance is symmetric, positive definite, and satisfies the triangle inequality:
    - $D(X, Y) = D(Y, X)$ (symmetric)
    - $D(X, Y) \geq 0$ (non-negativity)
    - $D(X, Y) = 0$ iff $X = Y$ (D(X,X)=0 reflexive)
    - $D(X, Y) + D(Y, Z) \geq D(X, Z)$ (triangle inequality)
Two unidimensional metric spaces

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Y</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Z</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>att</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>Y</td>
<td>3</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Z</td>
<td>8</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>att</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>
Multidimensional spaces using alternative metrics

Euclidian

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Y</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Z</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>W</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

City block

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>Y</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Z</td>
<td>7</td>
<td>4</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>W</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
A non metric space

<table>
<thead>
<tr>
<th>att</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Y</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>att</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
Multidimensional scaling

• Given a n * n distance matrix, is it possible to represent the data in a k dimensional space?
• How well does that model fit?
• How sensitive is the model to transformations of the original distances?
• Need to find distances
  • absolute distance between pairs
  • ranks of distances between pairs of pairs
## Distances between US cities

<table>
<thead>
<tr>
<th></th>
<th>ATL</th>
<th>BOS</th>
<th>ORD</th>
<th>DCA</th>
<th>DEN</th>
<th>LAX</th>
<th>MIA</th>
<th>JFK</th>
<th>SEA</th>
<th>SFO</th>
<th>MSY</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATL</td>
<td>0</td>
<td>934</td>
<td>585</td>
<td>542</td>
<td>1209</td>
<td>1942</td>
<td>605</td>
<td>751</td>
<td>2181</td>
<td>2139</td>
<td>424</td>
</tr>
<tr>
<td>BOS</td>
<td>934</td>
<td>0</td>
<td>853</td>
<td>392</td>
<td>1769</td>
<td>2601</td>
<td>1252</td>
<td>183</td>
<td>2492</td>
<td>2700</td>
<td>1356</td>
</tr>
<tr>
<td>ORD</td>
<td>585</td>
<td>853</td>
<td>0</td>
<td>598</td>
<td>918</td>
<td>1748</td>
<td>1187</td>
<td>720</td>
<td>1736</td>
<td>1857</td>
<td>830</td>
</tr>
<tr>
<td>DCA</td>
<td>542</td>
<td>392</td>
<td>598</td>
<td>0</td>
<td>1493</td>
<td>2305</td>
<td>922</td>
<td>209</td>
<td>2328</td>
<td>2442</td>
<td>964</td>
</tr>
<tr>
<td>DEN</td>
<td>1209</td>
<td>1769</td>
<td>918</td>
<td>1493</td>
<td>0</td>
<td>836</td>
<td>1723</td>
<td>1636</td>
<td>1023</td>
<td>951</td>
<td>1079</td>
</tr>
<tr>
<td>LAX</td>
<td>1942</td>
<td>2601</td>
<td>1748</td>
<td>2305</td>
<td>836</td>
<td>0</td>
<td>2345</td>
<td>2461</td>
<td>957</td>
<td>341</td>
<td>1679</td>
</tr>
<tr>
<td>MIA</td>
<td>605</td>
<td>1252</td>
<td>1187</td>
<td>922</td>
<td>1723</td>
<td>2345</td>
<td>0</td>
<td>1092</td>
<td>2733</td>
<td>2594</td>
<td>669</td>
</tr>
<tr>
<td>JFK</td>
<td>751</td>
<td>183</td>
<td>720</td>
<td>209</td>
<td>1636</td>
<td>2461</td>
<td>1092</td>
<td>0</td>
<td>2412</td>
<td>2577</td>
<td>1173</td>
</tr>
<tr>
<td>SEA</td>
<td>2181</td>
<td>2492</td>
<td>1736</td>
<td>2328</td>
<td>1023</td>
<td>957</td>
<td>2733</td>
<td>2412</td>
<td>0</td>
<td>681</td>
<td>2101</td>
</tr>
<tr>
<td>SFO</td>
<td>2139</td>
<td>2700</td>
<td>1857</td>
<td>2442</td>
<td>951</td>
<td>341</td>
<td>2594</td>
<td>2577</td>
<td>681</td>
<td>0</td>
<td>1925</td>
</tr>
<tr>
<td>MSY</td>
<td>424</td>
<td>1356</td>
<td>830</td>
<td>964</td>
<td>1079</td>
<td>1679</td>
<td>669</td>
<td>1173</td>
<td>2101</td>
<td>1925</td>
<td>0</td>
</tr>
</tbody>
</table>
## Multidimensional Scaling

<table>
<thead>
<tr>
<th>City</th>
<th>Dimension 1</th>
<th>Dimension 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATL</td>
<td>-571</td>
<td>248</td>
</tr>
<tr>
<td>BOS</td>
<td>-1061</td>
<td>-548</td>
</tr>
<tr>
<td>ORD</td>
<td>-264</td>
<td>-251</td>
</tr>
<tr>
<td>DCA</td>
<td>-861</td>
<td>-211</td>
</tr>
<tr>
<td>DEN</td>
<td>616</td>
<td>10</td>
</tr>
<tr>
<td>LAX</td>
<td>1370</td>
<td>376</td>
</tr>
<tr>
<td>MIA</td>
<td>-959</td>
<td>708</td>
</tr>
<tr>
<td>JFK</td>
<td>-970</td>
<td>-389</td>
</tr>
<tr>
<td>SEA</td>
<td>1438</td>
<td>-607</td>
</tr>
<tr>
<td>SFO</td>
<td>1563</td>
<td>88</td>
</tr>
<tr>
<td>MSY</td>
<td>-301</td>
<td>577</td>
</tr>
</tbody>
</table>

Code:

```r
cmdscale(cities)
round(cmdscale(cities),0)
```
Spatial representation

> cit <- cmdscale(cities)
> plot(cit, typ="n")
> text(cit, rownames(cit))
A more familiar map

> plot(-cit,typ="n",main="MDS of cities")
> text(-cit,rownames(cit))
Compare with classic solution
MultiDimensional Scaling of US cities

Cities:
- ATL
- BOS
- DEN
- DCA
- JFK
- LAX
- MIA
- MSY
- Ord
- SFO
- SEA
- SFO
R code for MDS

- http://personality-project.org/r/mds.html
- compare cmdscale (metric mds) with isoMDS (non-metric scaling)
- ALSCAL and KYST are standard packages in SPSS
- Individual Differences models of MDS include INDSCAL and INDIFF
- ALSCAL is now available in R
Metric scaling of 28 European cities

loc <- cmdscale(eurodist)
x <- loc[,1]
y <- -loc[,2]
plot(x, y, type="n", xlab="", ylab="", main="cmdscale(eurodist)")
text(x, y, names(eurodist), cex=0.8)
Types of data collected vs. types of questions asked

- Ask Si about
  - O
  - O x O
- infer
  - O x O
  - (O x O) x (O x O)
- S x O

<table>
<thead>
<tr>
<th></th>
<th>S₁</th>
<th>...</th>
<th>Sₙ</th>
<th>O₁</th>
<th>...</th>
<th>Oₙ</th>
</tr>
</thead>
<tbody>
<tr>
<td>S₁</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sₙ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O₁</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Oₙ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• $O = \{\text{Stimulus Objects}\}$  $S = \{\text{Subjects}\}$

• $O = \{o_1, o_2, \ldots, o_i, \ldots, o_n\}$

• $S = \{s_1, s_2, \ldots, s_i, \ldots, s_m\}$

• $S \times O = \{(s_1, o_1), (s_i, o_j), \ldots, (s_m, o_n)\}$

• $O \times O = \{(o_1, o_1), (o_i, o_j), \ldots, (o_n, o_n)\}$

• Types of Comparisons:
  - Order $s_i < o_j$ (aptitudes or amounts)
  - Proximities $|s_i - o_j| < d$ (preferences)
## Coombs typology of data

<table>
<thead>
<tr>
<th>Single Stimuli</th>
<th>Pairs of Dyads</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Single Dyads</strong></td>
<td><strong>Pairs of Dyads</strong></td>
</tr>
<tr>
<td>Measurement $(S^*O)$</td>
<td>Preferential choice $(S^*O) * (S^*O)$</td>
</tr>
<tr>
<td>$s_i &lt; o_j$ (Abilities)</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>s_i - o_j</td>
</tr>
<tr>
<td>Scaling of stimuli $(O * O)$</td>
<td>Multidimensional Scaling $(O^*O) * (O^*O)$</td>
</tr>
<tr>
<td>$o_i &lt; o_j$</td>
<td>$</td>
</tr>
</tbody>
</table>

Individual differences in Multidimensional Scaling $S * (O^*O) * (O^*O)$
Coombs’ typology

- O x O  \((o_i < o_j)\)  Scaling
- \((OxO) \times (OxO)\) \(|o_i - o_j| < |o_k - o_l|\) MDS
- S x O  (two types of comparisons)
  - \((s_i < o_j)\)  measurement of ability
  - \(|s_i - o_j| < d\)  measurement of attitude
- \((SxO) \times (SxO)\) preferential choice
  - \(|s_i - o_j| < |s_k - o_l|\)  or  \(|s_i - o_j| < |s_i - o_l|\)
Preferential Choice and Unfolding
\((S \ast O) \ast (S \ast O)\)

Comparison of the distance of subject to an item versus another subject to another item:
\[|s_i - o_j| < |s_k - o_l|\]

Do you like broccoli more than I like spinach?
Or more typically: do you like broccoli more than you like spinach? \[|s_i - o_j| < |s_i - o_l|\]

Preferential choice and Unfolding \((S \ast O) \ast (S \ast O)\)
Preferential Choice: Individual (I) scales

• Question asked an individual:
  – Do you prefer object j to object k?

• Model of answer:
  – Something is preferred to something else if if it “closer” in the attribute space or on a particular attribute dimension
  – Individual has an “Ideal point” on the attribute.
  – Objects have locations along the same attribute
  – $|s_i - o_j| < |s_i - o_k|$

  – The I scale is the individual’s rank ordering of preferences
Preferential Choice: J scales

- Individual preferences can give information about object to object distances that are true for multiple people.
- Locate people in terms of their I scales along a common J scale.
Preferential Choice: free choice

- If you had complete freedom of choice, how many children would you like to have? _X_
- If you could not have that many, what would your second choice be? _Y_
- Third choice? _Z_
- Fourth choice? -W-
- Fifth choice? _V_
Preferential Choice: forced choice

1. If you had complete freedom of choice, how many children would you like to have? _X_
2. If you could not have X, would you rather have X +1 or X-1 (Y).
3. If could not have X or Y, would you rather have (min(X,Y)-1) or max (X,Y)+1. (Z)
4. If you could have X, Y or Z, would you rather have min(X,Y,Z)-1 or max (X,Y, Z)+1
5. Repeat (4) until either 0 or 5
Preferential choice-underlying model

• On a scale from 0 to 100, if 0 means having 0 children, and 100 means having 5 children, please assign the relative location of 1, 2, 3, and 4 children.

• On this same scale, please give your preferences for having 0, 1, 2, 3, 4, or 5 children.
Questions about Scale Ratings

• Do subjects understand instructions?
• Can people give accurate representations of scale value to objects?
• Can we find subject locations in preferential space?
Alternative Joint scales

- Accelerating
- Decelerating
- Equal interval
4 Individual scales from the accelerating Joint scale
Individual scales from the deaccelerating Joint scale
Individual scales from two Joint scales
Unfolding of Preferences

• Consider the I scale 234105
• What information has this person given us?
• Unfold to give J scale
• Ideal point is closest to 2, furthest from 5.
• J scale of
  0 1 2 3 4 5
• Critical information: 2|3 occurs after 1|4
Joint scales, Points and Midpoints

Objects and Midpoints
Joint scales, Points and Midpoints

Accelerating scale

Objects and Midpoints

Object Values

0  20  40  60  80  100
0  1  2
0  1  2  3
0  1  2  3  4
0  1  2  3  4  5
I scales and midpoints

eexample 1

- Preference Orders:
  - Midpoints crossed
- (Individual Scales)

```
0 1 2 3 4
1 0 2 3 4 01
1 2 0 3 4 01 02
1 2 0 3 4 01 02 03
1 2 3 0 4 01 02 03 04
2 1 3 4 0 01 02 03 04 12
2 3 1 4 0 01 02 03 04 12 13
2 3 4 1 0 01 02 03 04 12 13 14
3 2 4 1 0 01 02 03 04 12 13 14 23
3 4 2 1 0 01 02 03 04 12 13 14 23 24
4 3 2 1 0 01 02 03 04 12 13 14 23 24 34
```
I scales and Midpoints: Example 2

- Preference Orders: Midpoints crossed
- (Individual Scales)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Distance information from midpoints

• Let \( x|y \) mean midpoint of \( x \) and \( y \) then the ordering of the midpoints provides information.

Consider:

• \( 1 \quad 1|4 \quad 4 \)

• \( 2 \quad 2|3 \quad 3 \quad \) vs

• \( 2 \quad 2|3 \quad 3 \)

• Midpoint orders imply distance information.

• If \( 2|3 < 1|4 \) then \( (12) < (34) \)

• If \( 2|3 > 1|4 \) then \( (12) > (34) \)
From midpoints to partial orders

- Data example 1
  - $0|3 < 1|2 \iff (01) > (23)$
  - $0|4 < 1|2 \iff (01) > (24)$
  - $0|4 < 1|3 \iff (01) > (34)$
  - $0|4 < 2|3 \iff (02) > (34)$
  - $1|4 < 2|3 \iff (12) > (34)$

- Partial Orders of distances
  - $(04) > (03) > (02) > (12) > (34)$
  - $(04) > (03) > (02) > (01) > (24) > (34)$
  - $(04) > (03) > (02) > (01) > (24) > (23)$
Family size Joint scale fitted from class data

- Alternative models:
  - Accelerating differences between children
  - De-accelerating differences
  - Equal spaced differences
### 405 data 1 scales and Midpoints

<table>
<thead>
<tr>
<th>Count</th>
<th>Scale</th>
<th>Midpoints &quot;crossed&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>012345</td>
<td></td>
</tr>
<tr>
<td></td>
<td>102345</td>
<td>01</td>
</tr>
<tr>
<td>3</td>
<td>210345</td>
<td>01 02 12</td>
</tr>
<tr>
<td>2</td>
<td>213045</td>
<td>01 02 12 03</td>
</tr>
<tr>
<td>3</td>
<td>213450</td>
<td>01 02 12 03 04 05</td>
</tr>
<tr>
<td></td>
<td>231045</td>
<td>01 02 12 03 13</td>
</tr>
<tr>
<td></td>
<td>231405</td>
<td>01 02 12 03 13 04</td>
</tr>
<tr>
<td></td>
<td>231450</td>
<td>01 02 12 03 13 04 05</td>
</tr>
<tr>
<td>3</td>
<td>321045</td>
<td>01 02 12 03 13 23</td>
</tr>
<tr>
<td>3</td>
<td>321450</td>
<td>01 02 12 03 13 23 04 05</td>
</tr>
<tr>
<td>2</td>
<td>324150</td>
<td>01 02 12 03 13 23 04 14 05</td>
</tr>
<tr>
<td></td>
<td>324510</td>
<td>01 02 12 03 13 23 04 14 05 15</td>
</tr>
<tr>
<td></td>
<td>342150</td>
<td>01 02 12 03 13 23 04 14 05 24</td>
</tr>
<tr>
<td>2</td>
<td>342510</td>
<td>01 02 12 03 13 23 04 14 05 24 15</td>
</tr>
<tr>
<td></td>
<td>342105</td>
<td>01 02 12 03 13 23 04 14 24</td>
</tr>
<tr>
<td></td>
<td>432105</td>
<td>01 02 12 03 13 23 04 14 24 34</td>
</tr>
<tr>
<td>3</td>
<td>432150</td>
<td>01 02 12 03 13 23 04 14 05 24 34</td>
</tr>
<tr>
<td></td>
<td>453210</td>
<td>01 02 12 03 13 23 04 14 05 24 15 34 25 35</td>
</tr>
<tr>
<td>2</td>
<td>543210</td>
<td>01 02 12 03 13 23 04 14 05 24 15 34 25 35 45</td>
</tr>
</tbody>
</table>
Partial metric information: Midpoint orders implies order of distance

<table>
<thead>
<tr>
<th>Midpoint order</th>
<th>distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 03</td>
<td>=&gt; (01)&lt;(23)</td>
</tr>
<tr>
<td>23 04</td>
<td>=&gt; (02)&lt;(34)</td>
</tr>
<tr>
<td>24 15</td>
<td>=&gt; (12)&lt;(45)</td>
</tr>
<tr>
<td>34 25</td>
<td>=&gt; (23)&lt;(45)</td>
</tr>
<tr>
<td>05 24</td>
<td>=&gt; (45)&lt;(02)</td>
</tr>
<tr>
<td>24 15</td>
<td>=&gt; (12)&lt;(45)</td>
</tr>
</tbody>
</table>
### Partial orders

<table>
<thead>
<tr>
<th></th>
<th>(01)&lt;(23)</th>
<th>(23)&lt;(45)</th>
<th>(45)&lt;(02)</th>
<th>(02)&lt;(34)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(01)&lt;(02)</td>
<td>(02)&lt;(34)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(01)&lt;(23)&lt;(45)&lt;(02)&lt;(34)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(12)&lt;(45)&lt;(02)&lt;(34)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Distances as deltas

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a+b</td>
<td>b+c+d</td>
<td>a+b+c</td>
<td>a+2b+c+d+e</td>
<td>a+b+c+d</td>
<td></td>
</tr>
</tbody>
</table>
3 a priori models
3 a priori + 1 fitted model
3 prior + 1 fitted models and implied midpoint orders
The range of acceptable fits versus a priori models
Measurement (S * O)

- Ordering of abilities: \( s_i < o_j \)
  Is a subject less than an object (i.e. does the subject miss the item).
  Order the items in terms of difficulty, and subjects in terms of ability.
  example: high jump or cognitive ability test

- Proximity of attitudes \( |s_i - o_j| < d \)
  Subject agrees (endorses) an item if \( d < \) some threshold
  Subject rejects the item if \( d > \) threshold
Error free models of ability: The Guttman scale
Error models of ability: Normal Ogive/Logistic of order

Cumulative normal density for 7 items
Measuring Attitudes: distance from ideal point => unfolding

Normal density for 7 items
### Coombs Typology of Data

<table>
<thead>
<tr>
<th></th>
<th>Single Dyads</th>
<th>Pairs of Dyads</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Single Stimuli</strong></td>
<td>Measurement $(S^O)$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$s_i &lt; o_j$ (Abilities)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>s_i - o_j</td>
</tr>
<tr>
<td><strong>Pairs of Stimuli</strong></td>
<td>Scaling of stimuli $(O^*O)$</td>
<td>MDS $(O^<em>O)^</em>(O^*O)$</td>
</tr>
<tr>
<td></td>
<td>$o_i &lt; o_j$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Scaling of people</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Individual differences in Multidimensional Scaling $S^*(O^<em>O)^</em>(O^*O)$</td>
</tr>
</tbody>
</table>
Psychometric Theory: A conceptual Syllabus

L1

X1
X2
X3

L2

X4
X5
X6

L3

X7
X8
X9

L4

Y1
Y2
Y3

L5

Y4
Y5
Y6

Y7
Y8
Measurement and scaling

Inferring latent values from observed values
Types of Scales: Inferences from observed variables to Latent variables

- Nominal
- Ordinal
- Interval
- Ratio

- Categories
  - Ranks \((x > y)\)
  - Differences
    - \(X - Y > W - V\)

- Equal intervals with a zero point =>
  - \(X/Y > W/V\)
Mappings and inferences

Latent variable (construct)

Observed data
Ordinal Scales

- Any monotonic transformation will preserve order
- Inferences from observed to latent variable are restricted to rank orders
- Statistics: Medians, Quartiles, Percentiles
Mappings and inferences

Latent variable (construct)

Observed data

L1
L2
L3
L4
O1
O2
O3
O4
Interval Scales

• Possible to infer the magnitude of differences between points on the latent variable given differences on the observed variable X is as much greater than Y as Z is from W
• Linear transformations preserve interval information
• Allowable statistics: Means, Variances
Mappings and inferences

Latent variable (construct)

Observed data

L4
L3
L2
L1

O1  O2  O3  O4
Ratio Scales

- Interval scales with a zero point
- Possible to compare ratios of magnitudes (X is twice as long as Y)
The search for appropriate scale

- Is today colder than yesterday? (ranks)
- Is the amount that today is colder than yesterday more than the amount that yesterday was colder than the day before? (intervals)
  - 50 F - 39 F < 68 F - 50 F
  - 10 C - 4 C < 20 C - 10 C
  - 283K - 277K < 293K -283K
- How much colder is today than yesterday?
  - (Degree days as measure of energy use)
  - K as measure of molecular energy
Gas consumption by degree days (65-T)

Heating demands (therms) by house and Degree Days

- Conventional
- Energy efficient no fireplace
- Energy efficient with fireplace
Latent and Observed Scores
The problem of scale

Much of our research is concerned with making inferences about latent (unobservable) scores based upon observed measures. Typically, the relationship between observed and latent scores is monotonic, but not necessarily (and probably rarely) linear. This leads to many problems of inference. The following examples are abstracted from real studies. The names have been changed to protect the guilty.
A leading research team in motivational and educational psychology was interested in the effect that different teaching techniques at various colleges and universities have upon their students. They were particularly interested in the effect upon writing performance of attending a very selective university, a less selective university, or a two year junior college.

A writing test was given to the entering students at three institutions in the Boston area. After one year, a similar writing test was given again. Although there was some attrition from each sample, the researchers report data only for those who finished one year. The pre and post test scores as well as the change scores were as shown below:
Effect of teaching upon performance

<table>
<thead>
<tr>
<th></th>
<th>Pretest</th>
<th>Posttest</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Junior College</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Non-selective university</td>
<td>5</td>
<td>27</td>
<td>22</td>
</tr>
<tr>
<td>Selective university</td>
<td>27</td>
<td>73</td>
<td>45</td>
</tr>
</tbody>
</table>

From these data, the researchers concluded that the quality of teaching at the very selective university was much better and that the students there learned a great deal more. They proposed to study the techniques used there in order to apply them to the other institutions.
Effect of Teaching upon Performance?
Another research team in motivational and educational psychology was interested in the effect that different teaching techniques at various colleges and universities have upon their students. They were particularly interested in the effect upon mathematics performance of attending a very selective university, a less selective university, or a two year junior college. A math test was given to the entering students at three institutions in the Boston area. After one year, a similar math test was given again. Although there was some attrition from each sample, the researchers report data only for those who finished one year. The pre and post test scores as well as the change scores were:

<table>
<thead>
<tr>
<th></th>
<th>Pretest</th>
<th>Posttest</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Junior College</td>
<td>27</td>
<td>73</td>
<td>45</td>
</tr>
<tr>
<td>Non-selective university</td>
<td>73</td>
<td>95</td>
<td>22</td>
</tr>
<tr>
<td>Selective university</td>
<td>95</td>
<td>99</td>
<td>4</td>
</tr>
</tbody>
</table>
Effect of Teaching upon

[Graph showing the effect of teaching on different types of education institutions: Selective, Non-Selective, and Junior College, with a comparison between Pre and Post teaching stages.]
A leading cognitive developmentalist believed that there is a critical stage for learning spatial representations using maps. Children younger than this stage are not helped by maps, nor are children older than this stage. He randomly assigned 3rd, 5th, and 7th grade students into two conditions (nested within grade), control and map use. Performance was measured on a task of spatial recall (children were shown toys at particular locations in a set of rooms and then asked to find them again later. Half the children were shown a map of the rooms before doing the task.

<table>
<thead>
<tr>
<th>Grade</th>
<th>No map</th>
<th>Maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>3rd</td>
<td>5</td>
<td>27</td>
</tr>
<tr>
<td>5th</td>
<td>27</td>
<td>73</td>
</tr>
<tr>
<td>7th</td>
<td>73</td>
<td>95</td>
</tr>
</tbody>
</table>
Spatial reasoning facilitated by maps at a critical age
Another cognitive developmentalist believed that there is a critical stage but that it appears earlier than previously thought. Children younger than this stage are not helped by maps, nor are children older than this stage. He randomly assigned 1\textsuperscript{st}, 3\textsuperscript{rd}, 5\textsuperscript{th}, and 7\textsuperscript{th} grade students into two conditions (nested within grade), control and map use. Performance was measured on a task of spatial recall (children were shown toys at particular locations in a set of rooms and then asked to find them again later). Half the children were shown a map of the rooms before doing the task.

<table>
<thead>
<tr>
<th>Grade</th>
<th>No map</th>
<th>Maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} grade</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3\textsuperscript{rd} grade</td>
<td>12</td>
<td>50</td>
</tr>
<tr>
<td>5\textsuperscript{th} grade</td>
<td>50</td>
<td>88</td>
</tr>
<tr>
<td>7\textsuperscript{th} grade</td>
<td>88</td>
<td>98</td>
</tr>
</tbody>
</table>
Spatial Reasoning is facilitated by map use at a
Cognitive-neuro psychologists believe that damage to the hippocampus affects long term but not immediate memory. As a test of this hypothesis, an experiment is done in which subjects with and without hippocampal damage are given an immediate and a delayed memory task. The results are impressive:

<table>
<thead>
<tr>
<th></th>
<th>Immediate</th>
<th>Delayed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hippocampus intact</td>
<td>98</td>
<td>88</td>
</tr>
<tr>
<td>Hippocampus damaged</td>
<td>95</td>
<td>73</td>
</tr>
</tbody>
</table>

From these results the investigator concludes that there are much larger deficits for the hippocampal damaged subjects on the delayed rather than the immediate task. The investigator believes these results confirm his hypothesis. Comment on the appropriateness of this conclusion.
Memory = f(hippocampal damage * temporal delay)
An investigator believes that caffeine facilitates attentional tasks such that require vigilance. Subjects are randomly assigned to conditions and receive either 0 or 4mg/kg caffeine and then do a vigilance task. Errors are recorded during the first 5 minutes and the last 5 minutes of the 60 minute task. The number of errors increases as the task progresses but this difference is not significant for the caffeine condition and is for the placebo condition.

<table>
<thead>
<tr>
<th></th>
<th>1st block</th>
<th>Last block</th>
</tr>
</thead>
<tbody>
<tr>
<td>Placebo (0 mg/kg)</td>
<td>8</td>
<td>40</td>
</tr>
<tr>
<td>Caffeine (4 mg/kg)</td>
<td>4</td>
<td>23</td>
</tr>
</tbody>
</table>
Errors = \textit{f}(\text{caffeine} \times \text{time on task})
Arousal is a fundamental concept in many psychological theories. It is thought to reflect basic levels of alertness and preparedness. Typical indices of arousal are measures of the amount of palmer sweating. This may be indexed by the amount of electricity that is conducted by the fingertips. Alternatively, it may be indexed (negatively) by the amount of skin resistance of the finger tips. The Galvanic Skin Response (GSR) reflects moment to moment changes, SC and SR reflect longer term, basal levels.

High skin conductance (low skin resistance) is thought to reflect high arousal.
Anxiety is thought to be related to arousal. The following data were collected by two different experimenters. One collected Resistance data, one conductance data.

<table>
<thead>
<tr>
<th></th>
<th>Resistance</th>
<th>Conductance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anxious</td>
<td>2, 2</td>
<td>.5, .5</td>
</tr>
<tr>
<td>Low anx</td>
<td>1, 5</td>
<td>1, .2</td>
</tr>
</tbody>
</table>

The means were

<table>
<thead>
<tr>
<th></th>
<th>Resistance</th>
<th>Conductance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anxious</td>
<td>2</td>
<td>.5</td>
</tr>
<tr>
<td>Low anx</td>
<td>3</td>
<td>.6</td>
</tr>
</tbody>
</table>

Experimenter 1 concluded that the low anxious had higher resistances, and thus were less aroused. But experimenter 2 noted that the low anxious had higher levels of skin conductance, and were thus more aroused.

How can this be?
Conductance = 1/Resistance

Non linear response and non equal Variances can lead to inconsistent Inferences about group differences

Average of High Anxious

Average Low Anxious
Performance and task difficulty

Performance as a function of Ability and Test Difficulty
Performance, ability, and task difficulty

<table>
<thead>
<tr>
<th>Latent Ability</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.00</td>
<td>0.12</td>
<td>0.05</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>-2.00</td>
<td>0.50</td>
<td>0.27</td>
<td>0.12</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>0.00</td>
<td>0.73</td>
<td>0.50</td>
<td>0.27</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>0.98</td>
<td>0.95</td>
<td>0.88</td>
<td>0.73</td>
<td>0.50</td>
</tr>
<tr>
<td>4.00</td>
<td>1.00</td>
<td>0.99</td>
<td>0.98</td>
<td>0.95</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Change from

-4 to -2
-2 to -0
0 to 2
2 to 4

0.38 0.22 0.10 0.04 0.02
0.38 0.46 0.38 0.22 0.10
0.10 0.22 0.38 0.46 0.38
0.02 0.04 0.10 0.22 0.38
Performance and Task Difficulty

Note that equal differences along the latent ability dimension result in unequal differences along the observed performance dimension. Compare particularly performance changes resulting from ability changes from -2 to 0 to 2 units.

This is taken from the standard logistic transformation used in Item Response Theory that maps latent ability and latent difficulty into observed scores. IRT attempts to estimate difficulty and ability from the observed patterns of performance.

\[
\text{Performance} = \frac{1}{1+\exp^{(\text{difficulty-ability})}}
\]
Decision making and the benefit of extreme selection ratios

- Typical traits are approximated by a normal distribution.
- Small differences in means or variances can lead to large differences in relative odds at the tails.
- Accuracy of decision/prediction is higher for extreme values.
- Do we infer trait mean differences from observing differences of extreme values?

(code for these graphs at
  - http://personality-project.org/r/extremescores.r)
Odds ratios as \( f(\text{mean difference, extremity}) \)

Difference = .5 sigma  Difference = 1.0 sigma

Normal density for two groups

Odds ratio of G1 vs G2

Normal density for two groups

Odds ratio of G1 vs G2
The effect of group differences on likelihood of extreme scores

Cumulative normal density for two groups

Odds ratio that person in Group exceeds x

Graph 1: Cumulative normal density for two groups

Graph 2: Cumulative normal density for two groups

Graph 3: Odds ratio that person in Group exceeds x

Graph 4: Odds ratio that person in Group exceeds x
The effect of differences of variance on odds ratios at the tails

Variance of two groups differ by 10%

Variance of two groups differs by 20%

Odds ratio of G1 vs G2

Odds ratio of G1 vs G2
Percentiles are not a linear metric and percentile odds are even worse!

- When comparing changes due to interventions or environmental trends, it is tempting to see how many people achieve a certain level (e.g., of educational accomplishment, or of obesity), but the magnitude of such changes is sensitive to starting points, particularly when using percentiles or even worse, odds of percentiles.
- Consider the case of obesity:
Obesity gets worse over time

• “Over the last 15 years, obesity in the US has doubled, going from one in 10 to one in five. But the prevalence of morbid obesity has quadrupled, meaning that the number of people 100 pounds overweight has gone from one in 200 to one in 50. And the number of people roughly 150 pounds overweight has increased by a factor of 5, spiraling from one in 2000 to one in 400.”

• “… The fact that super obesity is increasing faster than other categories of overweight suggests a strong environmental component (such as larger portions). If this were a strictly genetic predisposition, the numbers would rise only in proportion to the increase in other weight categories.” (Tufts Health Newsletter, Dec. 2003, p 2)
Is obesity getting worse for the super obese? - Seemingly

<table>
<thead>
<tr>
<th>Label</th>
<th>Definition</th>
<th>Odds</th>
<th>Change in Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obese</td>
<td>BMI = 30 40 lb for 5’5”</td>
<td>1/10 to 1/5</td>
<td>2</td>
</tr>
<tr>
<td>Morbid Obese</td>
<td>BMI = 40 100 lb</td>
<td>1/200 to 1/50</td>
<td>4</td>
</tr>
<tr>
<td>Super Obese</td>
<td>BMI = 50 150 lb</td>
<td>1/2000 to 1/400</td>
<td>5</td>
</tr>
</tbody>
</table>
Is obesity getting worse for the super obese?  -- No

<table>
<thead>
<tr>
<th>Label</th>
<th>Definition</th>
<th>Odds</th>
<th>Change in Odds</th>
<th>z score</th>
<th>Change in z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obese</td>
<td>BMI = 30</td>
<td>1/10 to 1/5</td>
<td>2</td>
<td>-1.28</td>
<td>-0.84</td>
</tr>
<tr>
<td>Morbid Obese</td>
<td>BMI = 40</td>
<td>1/200 to 1/50</td>
<td>4</td>
<td>-2.58</td>
<td>-2.05</td>
</tr>
<tr>
<td>Super Obese</td>
<td>BMI = 50</td>
<td>1/2000 to 1/400</td>
<td>5</td>
<td>-3.29</td>
<td>-2.81</td>
</tr>
</tbody>
</table>
Psychometric Theory: A conceptual Syllabus

L1
- X1
- X2
- X3
- X4
- X5
- X6
- X7
- X8
- X9

L2
- L1
- L3

L3
- L2
- L5

L4
- L1
- L5

L5
- L3
- L4

Y1
- Y2
- Y3
- Y4
- Y5
- Y6
- Y7
- Y8