Psychology 405: Psychometric Theory
reliability problem set
(with answers)

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Outline

Two tests

More tests

Validity

Answers
  Part I
  Answers, part 2

Validity:answers
For the next few problems, consider a 10 item test with two subtests, A and B. The average correlation in both subtests is .3, and the average item correlation between the two subtests is .2.

1. What is the variance of subtest A?
2. What is the covariance of A and B?
3. What is the correlation between A and B?
4. What is the variance of total test?
5. What is coefficient $\alpha$ for subtest A?
6. What is coefficient $\alpha$ for the entire test?
7. What is the worst split half reliability of the test? (Assuming that A and B represent the worst splits).
8. What is the correlation of an item in test A with test A?
9. What is the correlation of an item in test A with the total test?
10. What is $\omega_h$ for the entire test?
11. What is the correlation of test A with test B correcting for reliability

Now, we will do this again, but with the average correlation in test A = .2, in test B = .15, and the average inter-item correlation between the two tests is .1

12. What is the variance of subtest A?
13. What is the covariance of A and B?
14. What is the correlation between A and B?
15. What is the variance of total test?
16. What is coefficient $\alpha$ for subtest A?
17. What is coefficient $\alpha$ for the entire test?
18. What is the worst split half reliability of the test? (Assuming that A and B represent the worst splits).
19. What is the correlation of an item in test A with test A?
20. What is the correlation of an item in test A with the total test?
21. What is $\omega_h$ for the entire test?

22. What is the correlation of test A with test B correcting for reliability

There are 50 applicants for a position, you have reason to believe that 40% will succeed on the criterion. You accept a particular number (defined below) how many of the ones you accept will succeed if

23. You accept 40% randomly.

24. You accept 30% using a test with a validity coefficient of .5

25. You accept 40% using a test with a validity of .5?
Consider a 10 item test with two subtests, A and B. The average correlation in both subtests is .3, and the average item correlation between the two subtests is .2.

**Table:** Item correlations ⇒ Scale covariances and correlations

<table>
<thead>
<tr>
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<th>V1a</th>
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<th>V3a</th>
<th>V4a</th>
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Total Variance = 32
Sum of Item variances = 10.
\( \alpha_{AB} = .76 \)
\( \alpha_A = .68 \)
Average \( r = \bar{r} = (20 \times .3 + 20 \times .3 + 25 \times .2 + 25 \times .2) / 90 = .24 \)

\[
\alpha = \frac{V_t - \Sigma(v_i)}{V_t} = \frac{k}{k-1} = \frac{32 - 10}{32} = \frac{k \bar{r}}{1 + (k - 1)\bar{r}} = \frac{10 \times .244}{1 + 9 \times .244} = .76
\]
Table: Item correlations \[\iff\] Scale covariances and correlations

<table>
<thead>
<tr>
<th>Covariance</th>
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<td>A</td>
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</table>

\[
\alpha_{1-5} = \alpha = \frac{V_t - \sum (v_i)}{V_t} \frac{k}{k-1} = \frac{11 - 5}{11} \frac{5}{4} = \frac{k\bar{r}}{1 + (k-1)\bar{r}} = \frac{5 \times .3}{1 + 4 \times .3} = .68
\]

Worst split = \[
\frac{2r_{AB}}{1 + r_{AB}} = \frac{2 \times .45}{1 + .45} = .62
\]

Compare with \( \alpha_{1-10} = \frac{32 - 10}{32} \times \frac{10}{9} = .76 \)

which is greater.

Item whole correlations:

\[
r_{1,A} = \frac{\text{cov}_{1,1..5}}{\sqrt{V_1 V_A}} = \frac{2.2}{\sqrt{1 \times 11}} = .66
\]

\[
r_{1,(AB)} = \frac{\text{cov}_{1,1..10}}{\sqrt{V_1 V_{AB}}} = \frac{3.2}{\sqrt{1 \times 32}} = .56
\]

Correct for overlap: (replace \( v_i \) with \( \bar{r}r_{1,A} \))

\[
r_{1,(AB)} = \frac{\text{cov}_{1,1..10}}{\sqrt{V_1 V_{AB}}} = \frac{1.5}{\sqrt{1 \times 11}} = .45
\]

Adjust those for reliability: \[
\frac{.45}{\sqrt{.68}} = .55 \quad \text{and} \quad \frac{.44}{\sqrt{.68}} = .54
\]
Finding $\omega_h$ from the structure

1. Normally we want to find the factor loadings of $g$ to ask how much is recreated by the $g$ factor
2. But, we can do this by inspection: Items share a $g$ factor and group factors
3. Group factor variance $= 100 \times .2 = 20$
4. $\omega_h = \frac{\sigma^2_g}{\sigma^2_t} = \frac{20}{32} = .625$
5. In this particular case, this is the same as the worst split half
Now, we will do this again, but with the average correlation in test 
A = .2, in test B = .15, and the average inter-item correlation 
between the two tests is .1

**Table: Item correlations ⇒ Scale covariances and correlations**

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Total Variance = 22
Sum of Item variances = 10.

\[ \alpha_{AB} = .61 \quad \alpha_A = .55 \]

Average \( r = \bar{r} = (20 \times .2 + 20 \times .15 + 25 \times .1 + 25 \times .1)/90 = .133 \]

\[ \alpha = \frac{V_t - \sum (v_i)}{V_t} = \frac{k}{k-1} \frac{22 - 10}{22} \frac{10}{9} = \frac{k\bar{r}}{1 + (k-1)\bar{r}} = \frac{10 \times .133}{1 + 9 \times .133} = .61 \]

Worst split = \[ \frac{2r_{AB}}{1+r_{AB}} = \frac{2 \times .29}{1 + .29} = .45 \]

Compare with \( \alpha_{1-10} = \frac{22-10}{22} \times \frac{10}{9} = .61 \)

which is greater.
Finding $\omega_h$ from the structure

1. Normally we want to find the factor loadings of g to ask how much is recreated by the g factor
2. But, we can do this by inspection: Items share a g factor and group factors
3. Group factor variance $= 100 \times .1 = 10$
4. $\omega_h = \frac{\sigma_g^2}{\sigma_t^2} = \frac{10}{22} = .45$
5. In this particular case, this is the same as the worst split half
There are 50 applicants for a position, you have reason to believe that 40% will succeed on the criterion. You accept a particular number (defined below) how many of the ones you accept will succeed if

23. You accept 40% randomly.
24. You accept 30% using a test with a validity coefficient of .5
25. You accept 40% using a test with a validity of .5?
The four outcomes of a decision

Table: The four outcomes of a decision. Subjects above a particular score on the decision axes are accepted, those below are rejected. Similarly, the criterion of success is such that those above a particular value are deemed to have succeed, those below that value to have failed. All numbers are converted into percentages of the total.

<table>
<thead>
<tr>
<th>Decision = Predicted Outcome</th>
<th>Accept</th>
<th>Reject</th>
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</thead>
<tbody>
<tr>
<td>Success</td>
<td>Valid Positive (VP)</td>
<td>False Negative (FN)</td>
</tr>
<tr>
<td>Failure</td>
<td>False Positive (FP)</td>
<td>Valid Negative (VN)</td>
</tr>
</tbody>
</table>

Selection Rate (SR) = 1-Selection Rate (1-SR)

Accuracy = Valid Positive + Valid Negative

Sensitivity = Valid Positive / (Valid Positive + False Negative)

Specificity = Valid Negative / (Valid Negative + False Positive)

\[
\text{Phi} = \left( \frac{VP - BR \cdot SR}{\sqrt{BR(1 - BR) \cdot SR \cdot (1 - SR)}} \right)
\]

\[
VP = BR \cdot SR + \phi \cdot \sqrt{BR(1 - BR) \cdot SR \cdot (1 - SR)}
\]
Finding predictions

23. You accept 40% randomly. \(.4 \times .4 \times N = 16\%\) of 50 = 8

24. You accept 30% using a test with a validity coefficient of .5?
   - \(BR = .4\)
   - \(SR = .3\)
   - \(\phi = .5\)
   - \(VP = BR \times SR + \phi \times \sqrt{BR(1 - BR) \times SR(1 - SR)} = .4 \times .3 + .5 \times \sqrt{.4 \times .6 \times .3 \times .7} = 23\%\) or \(50 \times .23 = 11.6\)

25. You accept 40% using a test with a validity of .5?
   - \(BR = .4\)
   - \(SR = .4\)
   - \(\phi = .5\)
   - \(VP = BR \times SR + \phi \times \sqrt{BR(1 - BR) \times SR(1 - SR)} = .4 \times .4 + .5 \times \sqrt{.4 \times .6 \times .4 \times .6} = 23\%\) or \(50 \times .28 = 14\)