Psychology 360: Personality Research
Psychometric Theory – Reliability Theory

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Outline: Part I: Classical Test Theory

Observed Variables

X Y

X_1 Y_1

X_2 Y_2

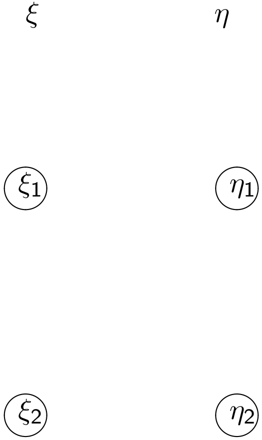
X_3 Y_3

X_4 Y_4

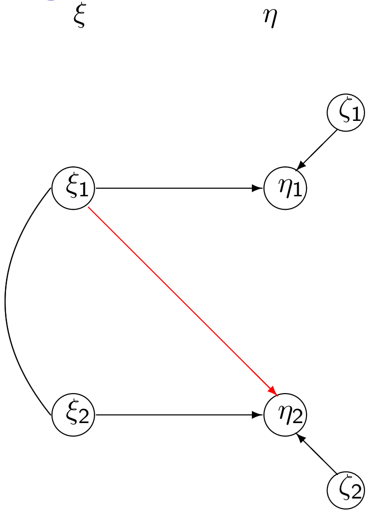
X_5 Y_5

X_6 Y_6

Latent Variables

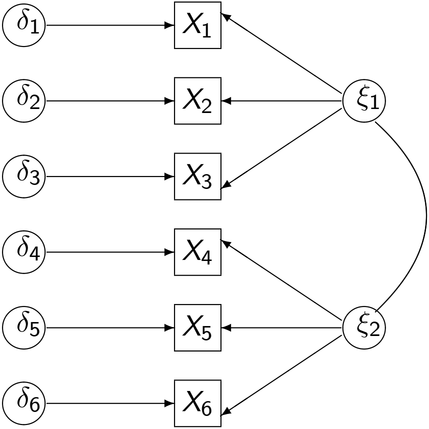


Theory: A regression model of latent variables

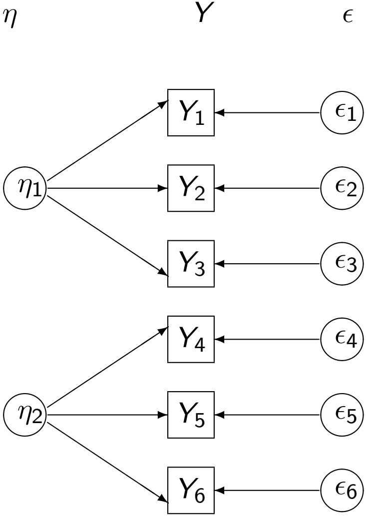


A measurement model for X – Correlated factors

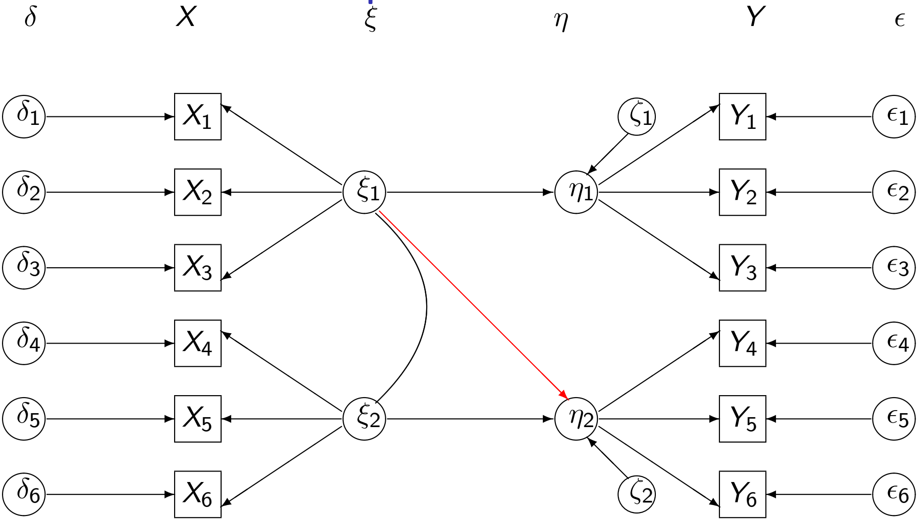
δ X ξ



A measurement model for Y - uncorrelated factors



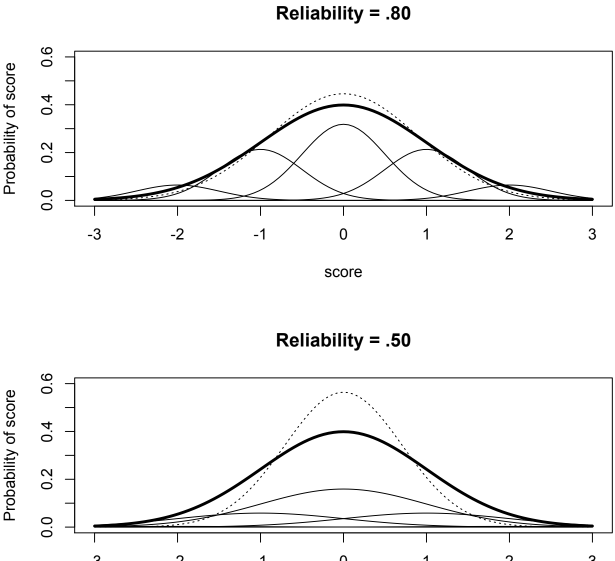
A complete structural model



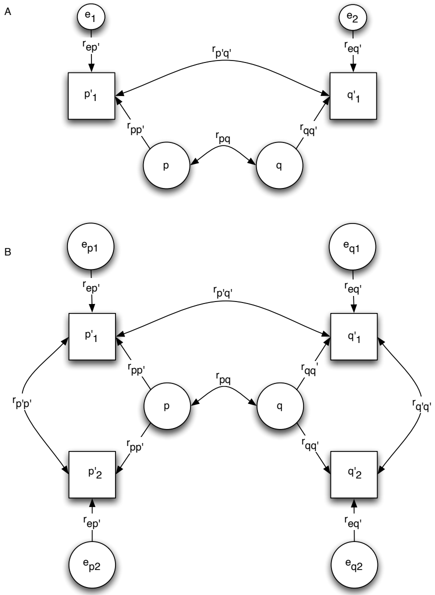
All data are befuddled with error

Now, suppose that we wish to ascertain the correspondence between a series of values, p, and another series, q. By practical observation we evidently do not obtain the true objective values, p and q, but only approximations which we will call p' and q'. Obviously, p' is less closely connected with q', than is p with q, for the first pair only correspond at all by the intermediation of the second pair; the real correspondence between p and q, shortly r_{pq} has been "attenuated" into $r_{p'q'}$ (Spearman, 1904, p 90).

All data are befuddled by error: Observed Score = True score + Error score



Spearman's parallell test theory



Classical True score theory

Let each individual score, x, reflect a true value, t, and an error value, e, and the expected score over multiple observations of x is t, and the expected score of e for any value of p is 0. Then, because the expected error score is the same for all true scores, the covariance of true score with error score (σ_{te}) is zero, and the variance of x, σ_x^2, is just

$$\sigma_x^2 = \sigma_t^2 + \sigma_e^2 + 2\sigma_{te} = \sigma_t^2 + \sigma_e^2.$$

Similarly, the covariance of observed score with true score is just the variance of true score

$$\sigma_{xt} = \sigma_t^2 + \sigma_{te} = \sigma_t^2$$

and the correlation of observed score with true score is

$$\rho_{xt} = \frac{\sigma_{xt}}{\sqrt{(\sigma_t^2 + \sigma_e^2)(\sigma_t^2)}} = \frac{\sigma_t^2}{\sqrt{\sigma_x^2 \sigma_t^2}} = \frac{\sigma_t}{\sigma_x}. \quad (1)$$

Classical Test Theory

By knowing the correlation between observed score and true score, ρ_{xt}, and from the definition of linear regression predicted true score, $\hat{t}$, for an observed x may be found from

$$\hat{t} = b_{t.x}x = \frac{\sigma_t^2}{\sigma_x^2}x = \rho_{xt}^2 x. \tag{2}$$

All of this is well and good, but to find the correlation we need to know either σ_t^2 or σ_e^2. The question becomes how do we find σ_t^2 or σ_e^2?.

Regression effects due to unreliability of measurement

Consider the case of air force instructors evaluating the effects of reward and punishment upon subsequent pilot performance. Instructors observe 100 pilot candidates for their flying skill. At the end of the day they reward the best 50 pilots and punish the worst 50 pilots.

- Day 1
  - Mean of best 50 pilots 1 is 75
  - Mean of worst 50 pilots is 25
- Day 2
  - Mean of best 50 has gone down to 65 (a loss of 10 points)
  - Mean of worst 50 has gone up to 35 (a gain of 10 points)
- It seems as if reward hurts performance and punishment helps performance.
- If there is no effect of reward and punishment, what is the expected correlation from day 1 to day 2?

Correcting for attenuation

> *To ascertain the amount of this attenuation, and thereby discover the true correlation, it appears necessary to make two or more independent series of observations of both p and q. (Spearman, 1904, p 90)*

Spearman's solution to the problem of estimating the true relationship between two variables, p and q, given observed scores p' and q' was to introduce two or more additional variables that came to be called *parallel tests*. These were tests that had the same true score for each individual and also had equal error variances. To Spearman (1904b p 90) this required finding "the average correlation between one and another of these independently obtained series of values" to estimate the reliability of each set of measures ($r_{p'p'}, r_{q'q'}$), and then to find

$$r_{pq} = \frac{r_{p'q'}}{\sqrt{r_{p'p'} r_{q'q'}}}. \quad (3)$$

Two parallel tests

The correlation between two parallel tests is the squared correlation of each test with true score and is the percentage of test variance that is true score variance

$$\rho_{xx} = \frac{\sigma_t^2}{\sigma_x^2} = \rho_{xt}^2. \tag{4}$$

Reliability is the fraction of test variance that is true score variance. Knowing the reliability of measures of p and q allows us to correct the observed correlation between p' and q' for the reliability of measurement and to find the unattenuated correlation between p and q.

$$r_{pq} = \frac{\sigma_{pq}}{\sqrt{\sigma_p^2 \sigma_q^2}} \tag{5}$$

and

$$r_{p'q'} = \frac{\sigma_{p'q'}}{\sqrt{\sigma_{p'}^2 \sigma_{q'}^2}} = \frac{\sigma_{p+e_1'}\sigma_{q+e_2'}}{\sqrt{\sigma_{p'}^2 \sigma_{q'}^2}} = \frac{\sigma_{pq}}{\sqrt{\sigma_{p'}^2 \sigma_{q'}^2}} \tag{6}$$

Modern "Classical Test Theory"

Reliability is the correlation between two *parallel tests* where tests are said to be parallel if for every subject, the true scores on each test are the expected scores across an infinite number of tests and thus the same, and the true score variances for each test are the same ($\sigma^2_{p'_1} = \sigma^2_{p'_2} = \sigma^2_{p'}$), and the error variances across subjects for each test are the same ($\sigma^2_{e'_1} = \sigma^2_{e'_2} = \sigma^2_{e'}$) (see Figure 19), (Lord & Novick, 1968; McDonald, 1999). The correlation between two parallel tests will be

$$\rho_{p'_1 p'_2} = \rho_{p'p'} = \frac{\sigma_{p'_1 p'_2}}{\sqrt{\sigma^2_{p'_1}\sigma^2_{p'_2}}} = \frac{\sigma^2_p + \sigma_{pe_1} + \sigma_{pe_2} + \sigma_{e_1 e_2}}{\sigma^2_{p'}} = \frac{\sigma^2_p}{\sigma^2_{p'}}. \quad (7)$$

Classical Test Theory

but from Eq 4,

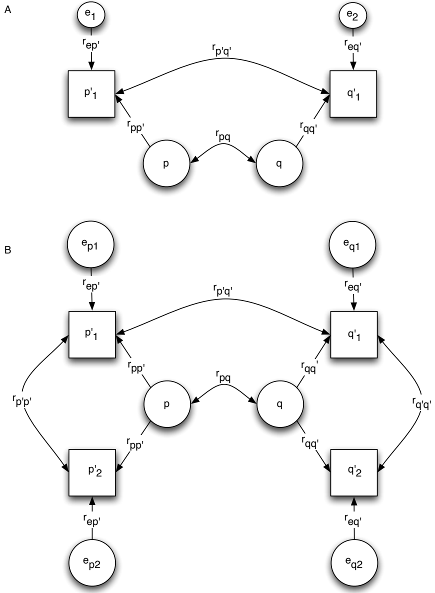
$$\sigma_p^2 = \rho_{p'p'}\sigma_{p'}^2 \quad (8)$$

and thus, by combining equation 5 with 6 and 8 the *unattenuated correlation* between p and q corrected for reliability is Spearman's equation 3

$$r_{pq} = \frac{r_{p'q'}}{\sqrt{r_{p'p'}r_{q'q'}}}. \quad (9)$$

As Spearman recognized, *correcting for attenuation* could show structures that otherwise, because of unreliability, would be hard to detect.

Spearman's parallell test theory



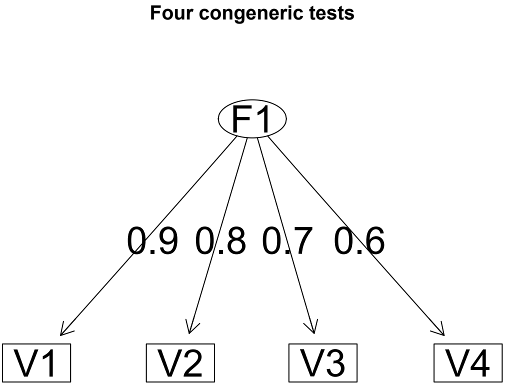
When is a test a parallel test?

But how do we know that two tests are parallel? For just knowing the correlation between two tests, without knowing the true scores or their variance (and if we did, we would not bother with reliability), we are faced with three knowns (two variances and one covariance) but ten unknowns (four variances and six covariances). That is, the observed correlation, $r_{p'_1 p'_2}$ represents the two known variances $s^2_{p'_1}$ and $s^2_{p'_2}$ and their covariance $s_{p'_1 p'_2}$. The model to account for these three knowns reflects the variances of true and error scores for p'_1 and p'_2 as well as the six covariances between these four terms. In this case of two tests, by defining them to be parallel with uncorrelated errors, the number of unknowns drop to three (for the true scores variances of p'_1 and p'_2 are set equal, as are the error variances, and all covariances with error are set to zero) and the (equal) reliability of each test may be found.

The problem of parallel tests

Unfortunately, according to this concept of parallel tests, the possibility of one test being far better than the other is ignored. Parallel tests need to be parallel by construction or assumption and the assumption of parallelism may not be tested. With the use of more tests, however, the number of assumptions can be relaxed (for three tests) and actually tested (for four or more tests).

Four congeneric tests – 1 latent factor



Observed variables and estimated parameters of a congeneric test

Observed correlations and modeled parameters

Variable	$Test_1$	$Test_2$	$Test_3$	$Test_4$
$Test_1$	$\sigma^2_{x_1} = \lambda_1\sigma^2_\theta + \epsilon^2_1$			
$Test_2$	$\sigma_{x_1x_2} = \lambda_1\sigma_\theta\lambda_2\sigma_\theta$	$\sigma^2_{x_2} = \lambda_2\sigma^2_\theta + \epsilon^2_2$		
$Test_3$	$\sigma_{x_1x_3} = \lambda_1\sigma_\theta\lambda_3\sigma_\theta$	$\sigma_{x_2x_3} = \lambda_2\sigma_\theta\lambda_3\sigma_\theta$	$\sigma^2_{x_3} = \lambda_3\sigma^2_\theta + \epsilon^2_3$	
$Test_4$	$\sigma_{x_1x_4} = \lambda_1\sigma_\theta\lambda_4\sigma_t$	$\sigma_{x_2x_4} = \lambda_2\sigma_\theta\lambda_4\sigma_\theta$	$\sigma_{x_3x_4} = \lambda_3\sigma_\theta\lambda_4\sigma_\theta$	$\sigma^2_{x_4} = \lambda_4\sigma^2_\theta + \epsilon^2_4$

We have a model of the observed variances and covariances in terms of the unknown parameters. We can solve these as a series of simultaneous equations. However, with just 2 tests we need to make some very strong assumptions ($\lambda_1 = \lambda_2$ and $\epsilon_1 = \epsilon_2$). With three tests, we can relax these assumptions need to assume either that $\lambda_1 = \lambda_2 = \lambda_3$ or $\epsilon_1 = \epsilon_2 = \epsilon_3$.

Observed variables and estimated parameters of a congeneric test

	V1	V2	V3	V4	V1	V2	V3	V
V1	s_1^2				$\lambda_1\sigma_t^2 + \sigma_{e_1}^2$			
V2	s_{12}	s_2^2			$\lambda_1\lambda_2\sigma_t^2$	$\lambda_2\sigma_t^2 + \sigma_{e_2}^2$		
V3	s_{13}	s_{23}	s_3^2		$\lambda_1\lambda_3\sigma_t^2$	$\lambda_2\lambda_3\sigma_t^2$	$\lambda_3\sigma_t^2 + \sigma_{e_3}^2$	
V4	s_{14}	s_{24}	s_{34}	s_4^2	$\lambda_1\lambda_4\sigma_t^2$	$\lambda_2\lambda_3\sigma_t^2$	$\lambda_3\lambda_4\sigma_t^2$	$\lambda_4\sigma_t^2$

Solve for the unknown parameters in terms of the known (observed) variances and covariances. We have a model of the observed variances and covariances in terms of the unknown parameters. We can solve these as a series of simultaneous equations. However, with just 2 tests we need to make some very strong assumptions ($\lambda_1 = \lambda_2$ and $\epsilon_1 = \epsilon_2$). With three tests, we can relax these assumptions need to assume either that $\lambda_1 = \lambda_2 = \lambda_3$ or $\epsilon_1 = \epsilon_2 = \epsilon_3$.

But what if we don't have three or more tests?

Unfortunately, with rare exceptions, we normally are faced with just one test, not two, three or four. How then to estimate the reliability of that one test? Defined as the correlation between a test and a test just like it, reliability would seem to require a second test. The traditional solution when faced with just one test is to consider the internal structure of that test. Letting reliability be the ratio of true score variance to test score variance (Equation 1), or alternatively, 1 - the ratio of error variance to true score variance, the problem becomes one of estimating the amount of error variance in the test. There are a number of solutions to this problem that involve examining the internal structure of the test. These range from considering the correlation between two random parts of the test to examining the structure of the items themselves.

Split halves

$$\Sigma_{XX'} = \begin{pmatrix} \mathbf{V_x} & \vdots & \mathbf{C_{xx'}} \\ \ldots & \ldots & \ldots \\ \mathbf{C_{xx'}} & \vdots & \mathbf{V_{x'}} \end{pmatrix} \quad (10)$$

and letting $V_{\mathbf{x}} = \mathbf{1V_x1'}$ and $C_{\mathbf{XX'}} = \mathbf{1C}_{XX'}\mathbf{1'}$ the correlation between the two tests will be

$$\rho = \frac{C_{xx'}}{\sqrt{V_x V_{x'}}}$$

But the variance of a test is simply the sum of the true covariances and the error variances:

$$V_{\mathbf{x}} = \mathbf{1V_x1'} = \mathbf{1C_t1'} + \mathbf{1V_e1'} = V_t + V_e$$

Split halves

and the structure of the two tests seen in Equation 10 becomes

$$\Sigma_{XX'} = \begin{pmatrix} \mathbf{V_X} = \mathbf{V_t} + \mathbf{V_e} & \vdots & \mathbf{C_{xx'}} = \mathbf{V_t} \\ \cdots & \cdots & \cdots \\ \mathbf{V_t} = \mathbf{C_{xx'}} & \vdots & \mathbf{V_{t'}} + \mathbf{V_{e'}} = \mathbf{V}_{X'} \end{pmatrix}$$

and because $\mathbf{V}_t = \mathbf{V}_{t'}$ and $\mathbf{V}_e = \mathbf{V}_{e'}$ the correlation between each half, (their reliability) is

$$\rho = \frac{C_{XX'}}{V_X} = \frac{V_t}{V_X} = 1 - \frac{V_e}{V_t}.$$

Split halves

The split half solution estimates reliability based upon the correlation of two random split halves of a test and the implied correlation with another test also made up of two random splits:

$$\Sigma_{XX'} = \left(\begin{array}{cc|cc} \mathbf{V_{x_1}} & \vdots\ \mathbf{C_{x_1x_2}} & \mathbf{C_{x_1x'_1}} & \vdots\ \mathbf{C_{x_1x'_2}} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \mathbf{C_{x_1x_2}} & \vdots\ \mathbf{V_{x_2}} & \mathbf{C_{x_2x'_1}} & \vdots\ \mathbf{C_{x_2x'_1}} \\ \hline \mathbf{C_{x_1x'_1}} & \vdots\ \mathbf{C_{x_2x'_1}} & \mathbf{V_{x'_1}} & \vdots\ \mathbf{C_{x'_1x'_2}} \\ \mathbf{C_{x_1x'_2}} & \vdots\ \mathbf{C_{x_2x'_2}} & \mathbf{C_{x'_1x'_2}} & \vdots\ \mathbf{V_{x'_2}} \end{array}\right)$$

Split halves

Because the splits are done at random and the second test is parallel with the first test, the expected covariances between splits are all equal to the true score variance of one split ($\mathbf{V_{t_1}}$), and the variance of a split is the sum of true score and error variances:

$$\Sigma_{XX'} = \left(\begin{array}{cc|cc} \mathbf{V_{t_1}} + \mathbf{V_{e_1}} & \mathbf{V_{t_1}} & \mathbf{V_{t_1}} & \mathbf{V_{t_1}} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{V_{t_1}} & \mathbf{V_{t_1}} + \mathbf{V_{e_1}} & \mathbf{V_{t_1}} & \mathbf{V_{t_1}} \\ \hline \mathbf{V_{t_1}} & \mathbf{V_{t_1}} & \mathbf{V_{t'_1}} + \mathbf{V_{e'_1}} & \mathbf{V_{t'_1}} \\ \mathbf{V_{t_1}} & \mathbf{V_{t_1}} & \mathbf{V_{t'_1}} & \mathbf{V_{t'_1}} + \mathbf{V_{e'_1}} \end{array}\right)$$

The correlation between a test made of up two halves with intercorrelation ($r_1 = V_{t_1}/V_{x_1}$) with another such test is

$$r_{xx'} = \frac{4V_{t_1}}{\sqrt{(4V_{t_1} + 2V_{e_1})(4V_{t_1} + 2V_{e_1})}} = \frac{4V_{t_1}}{2V_{t_1} + 2V_{x_1}} = \frac{4r_1}{2r_1 + 2}$$

and thus

The Spearman Brown Prophecy Formula

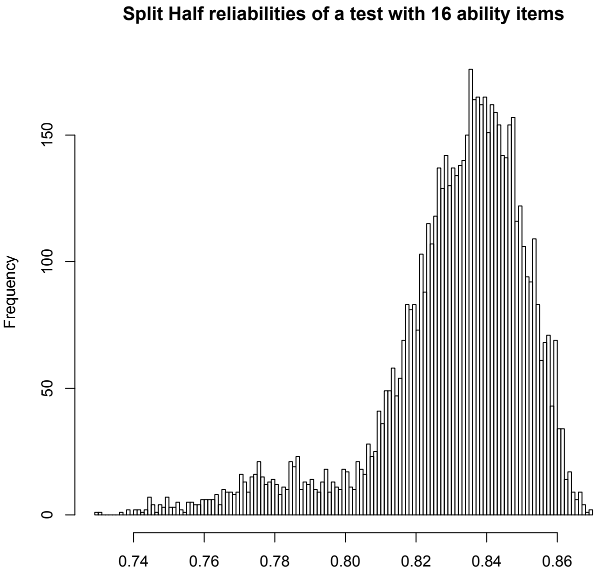
The correlation between a test made of up two halves with intercorrelation ($r_1 = V_{t_1}/V_{x_1}$) with another such test is

$$r_{xx'} = \frac{4V_{t_1}}{\sqrt{(4V_{t_1} + 2V_{e_1})(4V_{t_1} + 2V_{e_1})}} = \frac{4V_{t_1}}{2V_{t_1} + 2V_{x_1}} = \frac{4r_1}{2r_1 + 2}$$

and thus

$$r_{xx'} = \frac{2r_1}{1 + r_1} \tag{12}$$

6,435 possible eight item splits of the 16 ability items



Domain sampling

Other techniques to estimate the reliability of a single test are based on the *domain sampling* model in which tests are seen as being made up of items randomly sampled from a domain of items. Analogous to the notion of estimating characteristics of a population of people by taking a sample of people is the idea of sampling items from a universe of items.

Consider a test meant to assess English vocabulary. A person's vocabulary could be defined as the number of words in an unabridged dictionary that he or she recognizes. But since the total set of possible words can exceed 500,000, it is clearly not feasible to ask someone all of these words. Rather, consider a test of k words sampled from the larger domain of n words. What is the correlation of this test with the domain? That is, what is the correlation across subjects of test scores with their domain scores.?

Correlation of an item with the domain

First consider the correlation of a single (randomly chosen) item with the domain. Let the domain score for an individual be D_i and the score on a particular item, j, be X_{ij}. For ease of calculation, convert both of these to deviation scores. $d_i = D_i - \bar{D}$ and $x_{ij} = X_{ij} - \bar{X}_j$. Then

$$r_{x_j d} = \frac{cov_{x_j d}}{\sqrt{\sigma^2_{x_j} \sigma^2_d}}.$$

Now, because the domain is just the sum of all the items, the domain variance σ^2_d is just the sum of all the item variances and all the item covariances

$$\sigma^2_d = \sum_{j=1}^{n} \sum_{k=1}^{n} cov_{x_{jk}} = \sum_{j=1}^{n} \sigma^2_{x_j} + \sum_{j=1}^{n} \sum_{k \neq j} cov_{x_{jk}}.$$

Correlation of an item with the domain

Then letting $\bar{c} = \frac{\sum_{j=1}^{j=n} \sum_{k \neq j} cov_{x_{jk}}}{n(n-1)}$ be the average covariance and $\bar{v} = \frac{\sum_{j=1}^{j=n} \sigma^2_{x_j}}{n}$ the average item variance, the correlation of a randomly chosen item with the domain is

$$r_{x_j d} = \frac{\bar{v} + (n-1)\bar{c}}{\sqrt{\bar{v}(n\bar{v} + n(n-1)\bar{c})}} = \frac{\bar{v} + (n-1)\bar{c}}{\sqrt{n\bar{v}(\bar{v} + (n-1)\bar{c}))}}.$$

Squaring this to find the squared correlation with the domain and factoring out the common elements leads to

$$r^2_{x_j d} = \frac{(\bar{v} + (n-1)\bar{c})}{n\bar{v}}.$$

and then taking the limit as the size of the domain gets large is

$$\lim_{n \to \infty} r^2_{x_j d} = \frac{\bar{c}}{\bar{v}}. \tag{13}$$

That is, the squared correlation of an average item with the domain is the ratio of the average interitem covariance to the average item variance. Compare the correlation of a test with true

Domain sampling – correlation of an item with the domain

$$\lim_{n \to \infty} r^2_{x_j d} = \frac{\bar{c}}{\bar{v}}. \tag{14}$$

That is, the squared correlation of an average item with the domain is the ratio of the average interitem covariance to the average item variance. Compare the correlation of a test with true score (Eq 4) with the correlation of an item to the domain score (Eq 14). Although identical in form, the former makes assumptions about true score and error, the latter merely describes the domain as a large set of similar items.

Correlation of a test with the domain

A similar analysis can be done for a test of length k with a large domain of n items. A k-item test will have total variance, V_k, equal to the sum of the k item variances and the k(k-1) item covariances:

$$V_k = \sum_{i=1}^{k} v_i + \sum_{i=1}^{k} \sum_{j \neq i}^{k} c_{ij} = k\bar{v} + k(k-1)\bar{c}.$$

The correlation with the domain will be

$$r_{kd} = \frac{cov_k d}{\sqrt{V_k V_d}} = \frac{k\bar{v} + k(n-1)\bar{c}}{\sqrt{(k\bar{v} + k(k-1)\bar{c})(n\bar{v} + n(n-1)\bar{c})}} = \frac{k(\bar{v} + (n-1)\bar{c})}{\sqrt{nk(\bar{v} + (k-1)\bar{c})(\bar{v} + (n-1)\bar{c})}}$$

Correlation of a test with the domain

Then the squared correlation of a k item test with the n item domain is

$$r_{kd}^2 = \frac{k(\bar{v} + (n-1)\bar{c})}{n(\bar{v} + (k-1)\bar{c})}$$

and the limit as n gets very large becomes

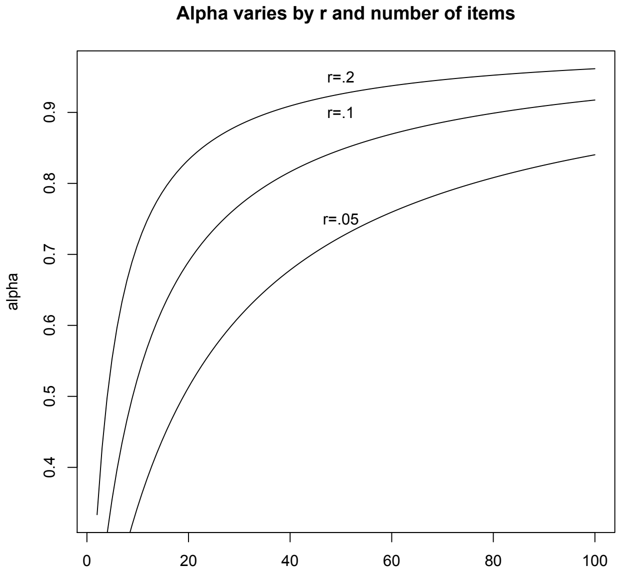
$$\lim_{n \to \infty} r_{kd}^2 = \frac{k\bar{c}}{\bar{v} + (k-1)\bar{c}}. \tag{15}$$

Coefficient α

Find the correlation of a test with a test just like it based upon the internal structure of the first test. Basically, we are just estimating the error variance of the individual items.

$$\alpha = r_{xx} = \frac{\sigma_t^2}{\sigma_x^2} = \frac{k^2 \frac{\sigma_x^2 - \sum \sigma_i^2}{k(k-1)}}{\sigma_x^2} = \frac{k}{k-1} \frac{\sigma_x^2 - \sum \sigma_i^2}{\sigma_x^2} \tag{16}$$

Alpha varies by the number of items and the inter item correlation



How to find α or KR20: Use your Frieden calculator

α is a function of Total Test Variance (V_X), sum of item variances (Σv_i) and number of items (n):

So, if you know how to add and subtract: $\alpha = \frac{V_X - \Sigma v_i}{V_x} \frac{n}{n-1}$

How to do modern statistics: Use R

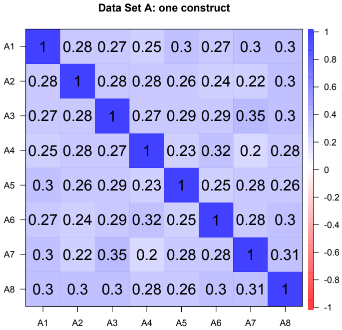
But we know more than addition and subtraction. We can do modern statistics and take advantage of computers.

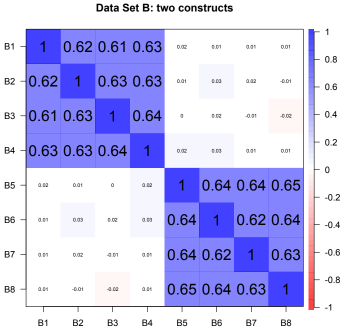
Consider two data sets, A and B. They look similar.

```
headTail(A)
      A1  A2  A3  A4  A5  A6  A7  A8
1      2   2   3   2   1   2   2   2
2      3   3   4   3   4   4   3   2
3      2   4   3   2   3   2   2   1
4      4   2   2   3   3   2   2   1
...  ... ... ... ... ... ... ... ...
997    4   4   4   3   2   4   1   4
998    2   2   2   2   2   1   3   2
999    4   4   4   3   4   3   4   5
1000   5   3   2   4   3   4   3   3

describe(A,skew=FALSE)
   vars    n mean   sd min max range   se
A1    1 1000 3.03 1.03   1   6     5 0.03
A2    2 1000 2.96 1.09   1   6     5 0.03
A3    3 1000 3.01 1.06   1   6     5 0.03
A4    4 1000 3.03 1.05   1   6     5 0.03
A5    5 1000 3.03 1.00   1   6     5 0.03
A6    6 1000 3.05 1.03   1   6     5 0.03
A7    7 1000 3.02 1.02   1   6     5 0.03
A8    8 1000 3.00 1.05   1   6     5 0.03

 alpha(A)

Call: alpha(x = A)

  raw_alpha std.alpha G6(smc) average_r med_r
      0.75      0.75    0.73      0.28   .28
  95% confidence boundaries
lower alpha upper
0.73 0.75 0.78
```

```
 headTail(B)
      B1  B2  B3  B4  B5  B6  B7  B8
1      2   3   1   2   1   2   1   1
2      3   4   3   4   3   2   3   3
3      3   4   4   4   1   3   2   3
4      2   3   3   2   3   2   3   3
...  ... ... ... ... ... ... ... ...
997    5   5   4   3   2   3   3   3
998    1   1   2   2   4   4   4   5
999    4   3   5   4   3   2   3   3
1000   3   3   3   3   3   3   3   3

describe(B,skew=FALSE)
   vars    n mean   sd min max range   se
B1    1 1000 3.07 1.04   1   6     5 0.03
B2    2 1000 3.02 1.00   1   6     5 0.03
B3    3 1000 3.02 1.02   1   6     5 0.03
B4    4 1000 3.04 1.01   1   6     5 0.03
B5    5 1000 3.00 1.03   1   6     5 0.03
B6    6 1000 3.02 0.99   1   6     5 0.03
B7    7 1000 3.02 1.02   1   6     5 0.03
B8    8 1000 3.01 0.99   1   6     5 0.03

alpha(B)

Call: alpha(x = B)

  raw_alpha std.alpha G6(smc) average_r  med_r
      0.75      0.75    0.84      0.28    .03
 95% confidence boundaries
 lower alpha upper
0.73 0.75 0.78
```

But they are actually quite different in their internal structure.



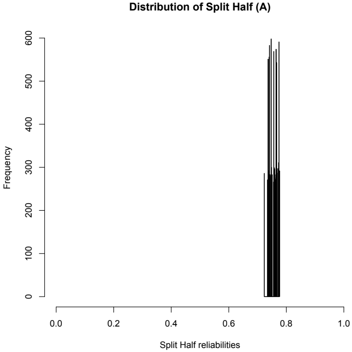


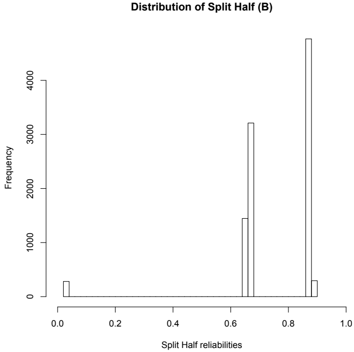
$\alpha = .75$
$\omega_h = .70$

$\alpha = .75$
$\omega_h = .03$

What is this thing called ω_h?

Distribution of all Split Half reliabilities differ






```
Split half reliabilities
Call: splitHalf(r = A, raw = TRUE)

Maximum split half reliability  =  0.78
Guttman lambda 6                =  0.73
Average split half reliability  =  0.75
Guttman lambda 3 (alpha)        =  0.75
Minimum split half reliability  =  0.72

 Quantiles of split half reliability
     2.5% 50% 97.5%
    0.72 0.76 0.78
```

```
Split half reliabilities
Call: splitHalf(r = B, raw = TRUE)

Maximum split half reliability =  0.88
Guttman lambda 6               =  0.84
Average split half reliability =  0.75
Guttman lambda 3 (alpha)       =  0.75
Minimum split half reliability =  0.03

 Quantiles of split half reliability
    2.5% 50% 97.5%
   0.03 0.87 0.88
```

Modeling the items and their correlations

1. An item represents at least four sources of variance
   - General factor variance ($\mathbf{g}$) which is common to all items on the test
   - Group factor variance ($\mathbf{f}$) which is common to some of the items on the test
   - Specific item variance ($\mathbf{s}$) reliable variance specific to an item
   - Residual (error) variance ($\mathbf{e}$), shared with nothing
2. Formally (in matrix notation) an item can be thought of as

$$\mathbf{x} = \mathbf{cg} + \mathbf{Af} + \mathbf{Ds} + \mathbf{e}.$$

   And a test ($\mathbf{X}$) made up of these items has a variance-covariance matrix ($\mathbf{C} = \mathbf{X'X}/N$) . The total test variance is just the sum of the individual elements of $\mathbf{C}$ and is

$$V_X = \mathbf{1C1'} = \mathbf{1'cc'1} + \mathbf{1'AA'1} + \mathbf{1'DD'1} + \mathbf{1'ee'1}$$

3. Our challenge: Estimate the variance associated with each of these sources.

Signal to Noise Ratio

The ratio of reliable variance to unreliable variance is known as the Signal/Noise ratio and is just

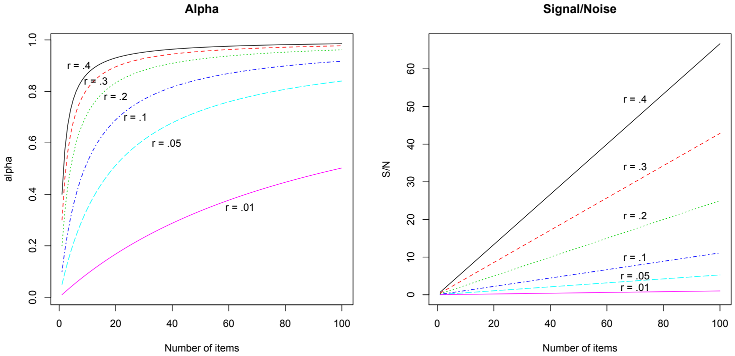
$$\frac{S}{N} = \frac{\rho^2}{1-\rho^2}$$

, which for the same assumptions as for α, will be

$$\frac{S}{N} = \frac{n\bar{r}}{1-\bar{r}}. \tag{17}$$

That is, the S/N ratio increases linearly with the number of items as well as with the average intercorrelation

Alpha vs signal/noise: and r and n



Find alpha using the alpha function

```
> alpha(bfi[16:20])

Reliability analysis
Call: alpha(x = bfi[16:20])

  raw_alpha std.alpha G6(smc) average_r mean  sd
       0.81      0.81     0.8      0.46   15 5.8

 Reliability if an item is dropped:
   raw_alpha std.alpha G6(smc) average_r
N1      0.75      0.75    0.70      0.42
N2      0.76      0.76    0.71      0.44
N3      0.75      0.76    0.74      0.44
N4      0.79      0.79    0.76      0.48
N5      0.81      0.81    0.79      0.51

 Item statistics
      n    r r.cor mean  sd
N1  990 0.81  0.78  2.8 1.5
N2  990 0.79  0.75  3.5 1.5
N3  997 0.79  0.72  3.2 1.5
N4  996 0.71  0.60  3.1 1.5
N5  992 0.67  0.52  2.9 1.6
```

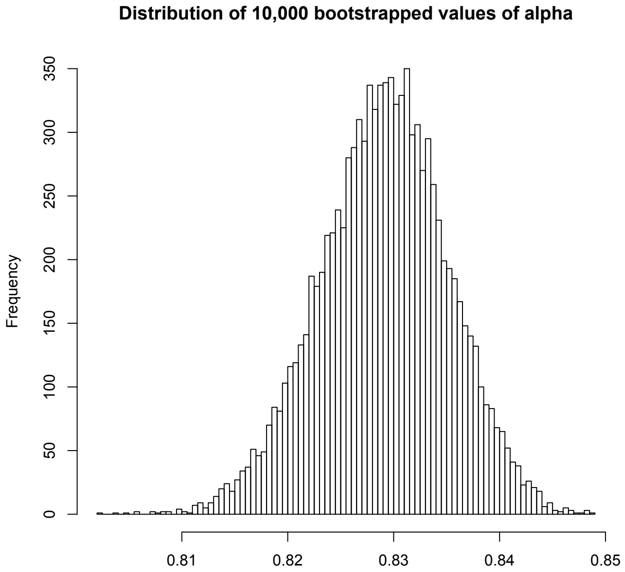
What if items differ in their direction?

```
> alpha(bfi[6:10],check.keys=FALSE)


 Reliability analysis
Call: alpha(x = bfi[6:10], check.keys = FALSE)

  raw_alpha std.alpha G6(smc) average_r mean   sd
      -0.28     -0.22    0.13    -0.038  3.8 0.58

 Reliability if an item is dropped:
   raw_alpha std.alpha G6(smc) average_r
C1    -0.430    -0.472  -0.020   -0.0871
C2    -0.367    -0.423  -0.017   -0.0803
C3    -0.263    -0.295   0.094   -0.0604
C4    -0.022     0.123   0.283    0.0338
C5    -0.028     0.022   0.242    0.0057

 Item statistics
      n    r r.cor  r.drop mean  sd
C1 2779 0.56  0.51  0.0354  4.5 1.2
C2 2776 0.54  0.51 -0.0076  4.4 1.3
C3 2780 0.48  0.27 -0.0655  4.3 1.3
C4 2774 0.20 -0.34 -0.2122  2.6 1.4
C5 2784 0.29 -0.19 -0.1875  3.3 1.6
```

But what if some items are reversed keyed?

```
alpha(bfi[6:10])

Reliability analysis
Call: alpha(x = bfi[6:10])

  raw_alpha std.alpha G6(smc) average_r mean   sd
      0.73      0.73    0.69      0.35  3.8 0.58
 Reliability if an item is dropped:
    raw_alpha std.alpha G6(smc) average_r
C1       0.69      0.70    0.64      0.36
C2       0.67      0.67    0.62      0.34
C3       0.69      0.69    0.64      0.36
C4-      0.65      0.66    0.60      0.33
C5-      0.69      0.69    0.63      0.36
 Item statistics
       n    r r.cor r.drop mean  sd
C1  2779 0.67  0.54   0.45  4.5 1.2
C2  2776 0.71  0.60   0.50  4.4 1.3
C3  2780 0.67  0.54   0.46  4.3 1.3
C4- 2774 0.73  0.64   0.55  2.6 1.4
C5- 2784 0.68  0.57   0.48  3.3 1.6
Warning message:  In alpha(bfi[6:10]) :
  Some items were negatively correlated with total scale and were
```

Bootstrapped confidence intervals for α



Guttman's alternative estimates of reliability

Reliability is amount of test variance that is not error variance. But what is the error variance?

$$r_{xx} = \frac{V_x - V_e}{V_x} = 1 - \frac{V_e}{V_x}. \quad (18)$$

$$\lambda_1 = 1 - \frac{tr(\mathbf{V_x})}{V_x} = \frac{V_x - tr(\mathbf{V}_x)}{V_x}. \quad (19)$$

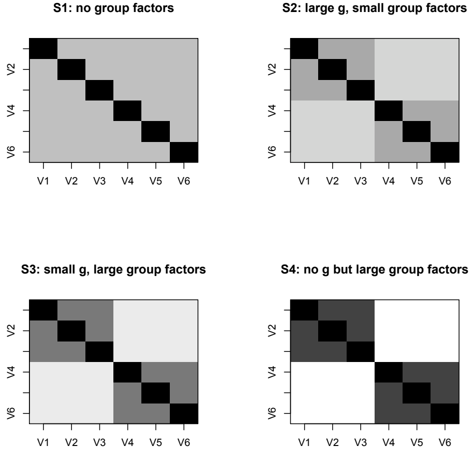
$$\lambda_2 = \lambda_1 + \frac{\sqrt{\frac{n}{n-1} C_2}}{V_x} = \frac{V_x - tr(\mathbf{V}_x) + \sqrt{\frac{n}{n-1} C_2}}{V_x}. \quad (20)$$

$$\lambda_3 = \lambda_1 + \frac{\frac{V_X - tr(\mathbf{V}_X)}{n(n-1)}}{V_X} = \frac{n\lambda_1}{n-1} = \frac{n}{n-1}\left(1 - \frac{tr(\mathbf{V})_x}{V_x}\right) = \frac{n}{n-1}\frac{V_x - tr(\mathbf{V}_x)}{V_x} = \alpha \quad (21)$$

$$\lambda_4 = 2\left(1 - \frac{V_{X_a} + V_{X_b}}{V_X}\right) = \frac{4c_{ab}}{V_x} = \frac{4c_{ab}}{V_{X_a} + V_{X_b} + 2c_{ab} V_{X_a} V_{X_b}}. \quad (22)$$

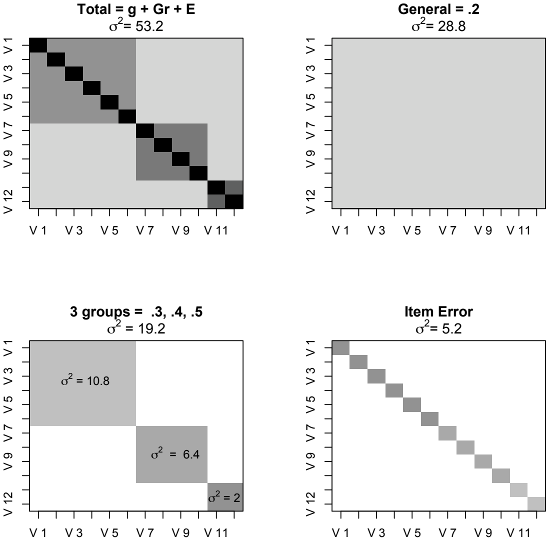
$$\lambda_6 = 1 - \frac{\sum e_j^2}{V_x} = 1 - \frac{\sum(1 - r_{smc}^2)}{V_x} \quad (23)$$

Four different correlation matrices, one value of α



1. The problem of group factors
2. If no groups, or many groups, α is ok

Decomposing a test into general, Group, and Error variance



1. Decompose total variance into general, group, specific, and error
2. $\alpha <$ total
3. $\alpha >$ general

Two additional alternatives to α: $\omega_{hierarchical}$ and $omega_{total}$

If a test is made up of a general, a set of group factors, and specific as well as error:

$$\mathbf{x} = \mathbf{cg} + \mathbf{Af} + \mathbf{Ds} + \mathbf{e} \tag{24}$$

then the communality of item$_j$, based upon general as well as group factors,

$$h_j^2 = c_j^2 + \sum f_{ij}^2 \tag{25}$$

and the unique variance for the item

$$u_j^2 = \sigma_j^2(1 - h_j^2) \tag{26}$$

may be used to estimate the test reliability.

$$\omega_t = \frac{\mathbf{1cc'1'} + \mathbf{1AA'1'}}{V_x} = 1 - \frac{\sum(1 - h_j^2)}{V_x} = 1 - \frac{\sum u^2}{V_x} \tag{27}$$

McDonald (1999) introduced two different forms for ω

$$\omega_t = \frac{\mathbf{1cc'1'} + \mathbf{1AA'1'}}{V_x} = 1 - \frac{\sum(1 - h_j^2)}{V_x} = 1 - \frac{\sum u^2}{V_x} \qquad (28)$$

and

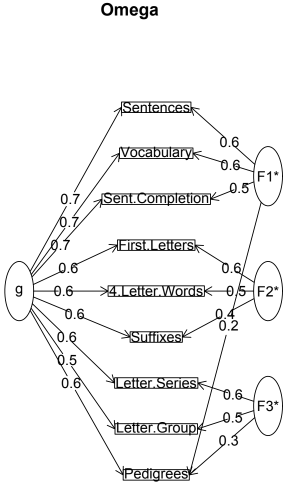
$$\omega_h = \frac{\mathbf{1cc'1}}{V_x} = \frac{(\sum \Lambda_i)^2}{\sum\sum R_{ij}}. \qquad (29)$$

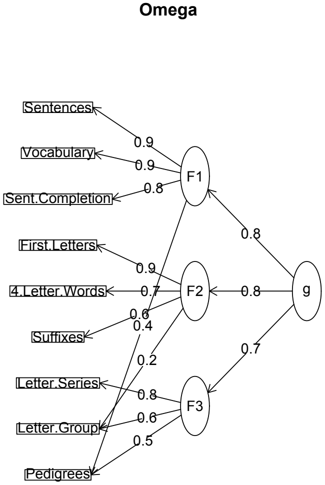
These may both be find by factoring the correlation matrix and finding the g and group factor loadings using the `omega` function.

Using omega on the Thurstone data set to find alternative reliability estimates

```
> lower.mat(Thurstone)
> omega(Thurstone)


                  Sntnc  Vcblr  Snt.C  Frs.L  4.L.W  Sffxs  Ltt.S  Pdgrs  Lt
Sentences         1.00
Vocabulary        0.83   1.00
Sent.Completion   0.78   0.78   1.00
First.Letters     0.44   0.49   0.46   1.00
4.Letter.Words    0.43   0.46   0.42   0.67   1.00
Suffixes          0.45   0.49   0.44   0.59   0.54   1.00
Letter.Series     0.45   0.43   0.40   0.38   0.40   0.29   1.00
Pedigrees         0.54   0.54   0.53   0.35   0.37   0.32   0.56   1.00
Letter.Group      0.38   0.36   0.36   0.42   0.45   0.32   0.60   0.45
1.00

Omega
Call: omega(m = Thurstone)
Alpha:                  0.89
G.6:                    0.91
Omega Hierarchical:     0.74
Omega H asymptotic:     0.79
Omega Total             0.93
```

Two ways of showing a general factor





omega function does a Schmid Leiman transformation

```
> omega(Thurstone, sl=FALSE)
```

```
Omega
Call: omega(m = Thurstone, sl = FALSE)
Alpha:                   0.89
G.6:                     0.91
Omega Hierarchical:      0.74
Omega H asymptotic:      0.79
Omega Total              0.93
Schmid Leiman Factor loadings greater than  0.2
                    g   F1*   F2*   F3*   h2   u2   p2
Sentences        0.71  0.57              0.82 0.18 0.61
Vocabulary       0.73  0.55              0.84 0.16 0.63
Sent.Completion  0.68  0.52              0.73 0.27 0.63
First.Letters    0.65        0.56        0.73 0.27 0.57
4.Letter.Words   0.62        0.49        0.63 0.37 0.61
Suffixes         0.56        0.41        0.50 0.50 0.63
Letter.Series    0.59              0.61  0.72 0.28 0.48
Pedigrees        0.58  0.23        0.34  0.50 0.50 0.66
Letter.Group     0.54              0.46  0.53 0.47 0.56
With eigenvalues of:
   g  F1*  F2*  F3*
3.58 0.96 0.74 0.71
```

Types of reliability

- Internal consistency
  - α
  - $\omega_{hierarchical}$
  - ω_{total}
  - β
- Intraclass
- Agreement
- Test-retest, alternate form
- Generalizability

- Internal consistency
  - `alpha`, `score.items`
  - `omega`
  - `iclust`
- `icc`
- `wkappa`, `cohen.kappa`
- `cor`
- `aov`

Alpha and its alternatives

- Reliability $= \frac{\sigma_t^2}{\sigma_x^2} = 1 - \frac{\sigma_e^2}{\sigma_x^2}$
- If there is another test, then $\sigma_t = \sigma_{t_1 t_2}$ (covariance of test X_1 with test $X_2 = C_{xx}$)
- But, if there is only one test, we can *estimate* σ_t^2 based upon the observed covariances within test 1
- How do we find σ_e^2 ?
- The worst case, (Guttman case 1) all of an item's variance is error and thus the error variance of a test X with variance-covariance C_x
    - $C_x = \sigma_e^2 = diag(C_x)$
    - $\lambda_1 = \frac{C_x - diag(C_x)}{C_x}$
- A better case (Guttman case 3, α) is that that the average covariance between the items on the test is the same as the average true score variance for each item.
    - $C_x = \sigma_e^2 = diag(C_x)$
    - $\lambda_3 = \alpha = \lambda_1 * \frac{n}{n-1} = \frac{(C_x - diag(C_x)) * n/(n-1)}{C_x}$

Guttman 6: estimating using the Squared Multiple Correlation

- Reliability $= \frac{\sigma_t^2}{\sigma_x^2} = 1 - \frac{\sigma_e^2}{\sigma_x^2}$
- Estimate true item variance as squared multiple correlation with other items
- $\lambda_6 = \frac{(C_x - diag(C_x) + \Sigma(smc_i))}{C_x}$
    - This takes observed covariance, subtracts the diagonal, and replaces with the squared multiple correlation
    - Similar to α which replaces with average inter-item covariance
- Squared Multiple Correlation is found by `smc` and is just $smc_i = 1 - 1/R_{ii}^{-1}$

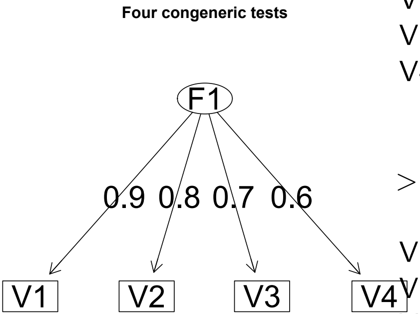
Alpha and its alternatives: Case 1: congeneric measures

First, create some simulated data with a known structure

```
> set.seed(42)
> v4 <- sim.congeneric(N=200,short=FALSE)
> str(v4)  #show the structure of the resulting object
List of 6
 $ model    : num [1:4, 1:4] 1 0.56 0.48 0.4 0.56 1 0.42 0.35 0.48 0.42 ...
  ..- attr(*, "dimnames")=List of 2
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
 $ pattern  : num [1:4, 1:5] 0.8 0.7 0.6 0.5 0.6 ...
  ..- attr(*, "dimnames")=List of 2
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
  .. ..$ : chr [1:5] "theta" "e1" "e2" "e3" ...
 $ r        : num [1:4, 1:4] 1 0.546 0.466 0.341 0.546 ...
  ..- attr(*, "dimnames")=List of 2
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
 $ latent   : num [1:200, 1:5] 1.371 -0.565 0.363 0.633 0.404 ...
  ..- attr(*, "dimnames")=List of 2
  .. ..$ : NULL
  .. ..$ : chr [1:5] "theta" "e1" "e2" "e3" ...
 $ observed: num [1:200, 1:4] -0.104 -0.251 0.993 1.742 -0.503 ...
  ..- attr(*, "dimnames")=List of 2
  .. ..$ : NULL
  .. ..$ : chr [1:4] "V1" "V2" "V3" "V4"
 $ N        : num 200
 - attr(*, "class")= chr [1:2] "psych" "sim"
```

A congeneric model

```
> f1 <- fa(v4$model)
> fa.diagram(f1)
```




```
> v4$model
     V1   V2   V3   V4
V1 1.00 0.56 0.48 0.40
V2 0.56 1.00 0.42 0.35
V3 0.48 0.42 1.00 0.30
V4 0.40 0.35 0.30 1.00
```

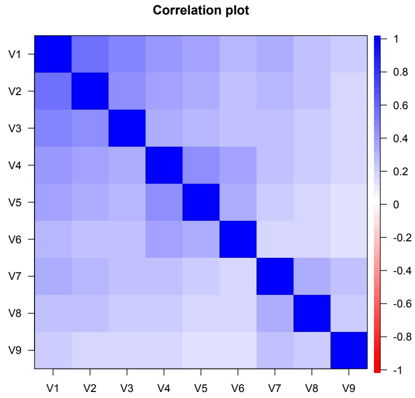
```
> round(cor(v4$observed),2
     V1   V2   V3   V4
V1 1.00 0.55 0.47 0.34
V2 0.55 1.00 0.38 0.30
V3 0.47 0.38 1.00 0.31
V4 0.34 0.30 0.31 1.00
```

Find α and related stats for the simulated data

```
> alpha(v4$observed)

 Reliability analysis
Call: alpha(x = v4$observed)

  raw_alpha std.alpha G6(smc) average_r   mean   sd
       0.71      0.72    0.67      0.39 -0.036 0.72

 Reliability if an item is dropped:
   raw_alpha std.alpha G6(smc) average_r
V1      0.59      0.60    0.50      0.33
V2      0.63      0.64    0.55      0.37
V3      0.65      0.66    0.59      0.40
V4      0.72      0.72    0.64      0.46

 Item statistics
     n    r r.cor r.drop   mean   sd
V1 200 0.80  0.72   0.60 -0.015 0.93
V2 200 0.76  0.64   0.53 -0.060 0.98
V3 200 0.73  0.59   0.50 -0.119 0.92
V4 200 0.66  0.46   0.40  0.049 1.09
```

A hierarchical structure

cor.plot(r9)

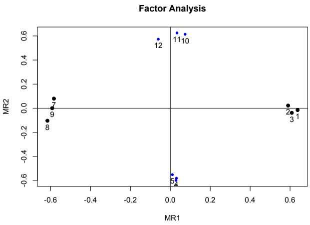



```
> set.seed(42)
> r9 <- sim.hierarchical()
> lower.mat(r9)


    V1   V2   V3   V4   V5   V6   V7
V8   V9
V1  1.00
V2  0.56 1.00
V3  0.48 0.42 1.00
V4  0.40 0.35 0.30 1.00
V5  0.35 0.30 0.26 0.42 1.00
V6  0.29 0.25 0.22 0.35 0.30 1.00
V7  0.30 0.26 0.23 0.24 0.20 0.17 1.00
V8  0.25 0.22 0.19 0.20 0.17 0.14 0.30 1.00
V9  0.20 0.18 0.15 0.16 0.13 0.11 0.24 0.20 1.(
```

α of the 9 hierarchical variables

```
> alpha(r9)

 Reliability analysis
Call: alpha(x = r9)

  raw_alpha std.alpha G6(smc) average_r
       0.76      0.76    0.76      0.26

 Reliability if an item is dropped:
   raw_alpha std.alpha G6(smc) average_r
V1      0.71      0.71    0.70      0.24
V2      0.72      0.72    0.71      0.25
V3      0.74      0.74    0.73      0.26
V4      0.73      0.73    0.72      0.25
V5      0.74      0.74    0.73      0.26
V6      0.75      0.75    0.74      0.27
V7      0.75      0.75    0.74      0.27
V8      0.76      0.76    0.75      0.28
V9      0.77      0.77    0.76      0.29

 Item statistics
      r r.cor
V1 0.72  0.71
V2 0.67  0.63
```

An example of two different scales confused as one

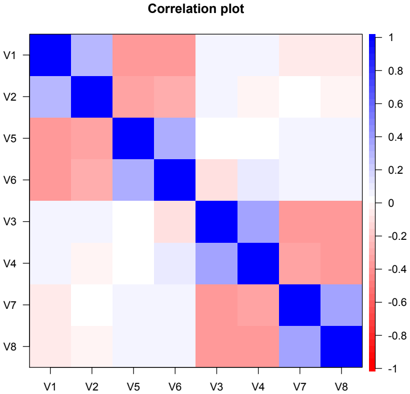



```
> set.seed(17)
> two.f <- sim.item(8)
> lower.mat(cor(two.f))
```

cor.plot(cor(two.f))

```
    V1    V2    V3    V4    V5    V6
V7    V8
V1   1.00
V2   0.29  1.00
V3   0.05  0.03  1.00
V4   0.03 -0.02  0.34  1.00
V5  -0.38 -0.35 -0.02 -0.01  1.00
V6  -0.38 -0.33 -0.10  0.06  0.33  1.00
V7  -0.06  0.02 -0.40 -0.36  0.03  0.04
1.00
V8  -0.08 -0.04 -0.39 -0.37  0.05  0.03
0.37  1.00
```

Rearrange the items to show it more clearly




```
> cor.2f <- cor(two.f)
> cor.2f  <- cor.2f[c(1:2,5:6,3:4,7:8),
          c(1:2,5:6,3:4,7:8)]
> lower.mat(cor.2f)
>cor.plot(cor.2f)
```

```
    V1    V2    V5    V6    V3    V4
V7    V8
V1   1.00
V2   0.29  1.00
V5  -0.38 -0.35  1.00
V6  -0.38 -0.33  0.33  1.00
V3   0.05  0.03 -0.02 -0.10  1.00
V4   0.03 -0.02 -0.01  0.06  0.34  1.00
V7  -0.06  0.02  0.03  0.04 -0.40 -0.36
1.00
V8  -0.08 -0.04  0.05  0.03 -0.39 -0.37
0.37  1.00
```

α of two scales confused as one

Note the use of the keys parameter to specify how some items should be reversed.

```
> alpha(two.f,keys=c(rep(1,4),rep(-1,4)))

 Reliability analysis
Call: alpha(x = two.f, keys = c(rep(1, 4), rep(-1, 4)))

  raw_alpha std.alpha G6(smc) average_r    mean   sd
      0.62      0.62    0.65      0.17 -0.0051 0.27

 Reliability if an item is dropped:
   raw_alpha std.alpha G6(smc) average_r
V1      0.59      0.58    0.61      0.17
V2      0.61      0.60    0.63      0.18
V3      0.58      0.58    0.60      0.16
V4      0.60      0.60    0.62      0.18
V5      0.59      0.59    0.61      0.17
V6      0.59      0.59    0.61      0.17
V7      0.58      0.58    0.61      0.17
V8      0.58      0.58    0.60      0.16

 Item statistics
     n    r r.cor r.drop   mean   sd
V1 500 0.54  0.44   0.33  0.063 1.01
V2 500 0.48  0.35   0.26  0.070 0.95
V3 500 0.56  0.47   0.36 -0.030 1.01
V4 500 0.48  0.37   0.28 -0.130 0.97
V5 500 0.52  0.42   0.31 -0.073 0.97
V6 500 0.52  0.41   0.31 -0.071 0.95
V7 500 0.53  0.44   0.34  0.035 1.00
V8 500 0.56  0.47   0.36  0.097 1.02
```

Score as two different scales

First, make up a keys matrix to specify which items should be scored, and in which way

```
> keys <- make.keys(nvars=8,keys.list=list(one=c(1,2,-5,-6),two=c(
> keys
     one two
[1,]   1   0
[2,]   1   0
[3,]   0   1
[4,]   0   1
[5,]  -1   0
[6,]  -1   0
[7,]   0  -1
[8,]   0  -1
```

Now score the two scales and find α and other reliability estimates

```
> score.items(keys,two.f)
Call: score.items(keys = keys, items = two.f)
(Unstandardized) Alpha:
       one two
alpha 0.68 0.7
Average item correlation:
           one  two
average.r 0.34 0.37
 Guttman 6* reliability:
          one  two
Lambda.6 0.62 0.64
Scale intercorrelations corrected for attenuation
 raw correlations below the diagonal, alpha on the diagonal
 corrected correlations above the diagonal:
     one  two
one 0.68 0.08
two 0.06 0.70
Item by scale correlations:
 corrected for item overlap and scale reliability
     one   two
V1   0.57  0.09
V2   0.52  0.01
V3   0.09  0.59
V4  -0.02  0.56
V5  -0.58 -0.05
V6  -0.57 -0.05
V7  -0.05 -0.58
V8  -0.09 -0.59
```

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